Abstracts

Free subgroups of arithmetic 3-manifold groups

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A group Γ is called *k-free* if any subgroup of Γ generated by k elements is free. We denote the maximal k for which Γ is k-free by $\mathcal{N}_{fr}(\Gamma)$. In [Gr87, Section 5.3.A], Gromov stated that $\mathcal{N}_{fr}(\Gamma)$ of a δ-hyperbolic group Γ is bounded below by an exponential function of the systole (or injectivity radius) of the associated quotient space M. The details of the proof were not given in [Gr87], they can be found in a later paper [Gr09, Section 2.4] where it is pointed out that the argument gives only a bound of the form $\epsilon r/\log(r)$, $r = \mathrm{sys}_1(M)$. Two other proofs of the growth of $\mathcal{N}_{fr}(\Gamma)$ when $\mathrm{sys}_1(M) \to \infty$ appear in [Ar06] and [KW03], but the quantitative bounds for $\mathcal{N}_{fr}(\Gamma)$ which can be deduced from these papers are weaker than the one above: Arzhantseva gives a bound of the form $cr^{1/3}$, Kapovich and Weidmann do not present an explicit estimate but the method of their paper would not allow to produce a considerably better bound. Thus so far Gromov's estimate appears to be the best available general quantitative result about $\mathcal{N}_{fr}(\Gamma)$. Although the difference between sub-linear and exponential growth is very large, in [Gr09, p. 763] Gromov conjectured that the true bound should be exponential. He pointed out that this is not known even for the fundamental groups of hyperbolic 3-manifolds. The main purpose of this talk is to discuss a result confirming Gromov's conjecture in this important special case.

Theorem 1. *Let M be an arithmetic hyperbolic 3-manifold defined by a quadratic form and $M_i \to M$ be a sequence of its congruence covers. Then*

$$\log \mathcal{N}_{fr}(\pi_1(M_i)) \gtrsim \frac{1}{3}\mathrm{sys}_1(M_i), \ \text{as } i \to \infty.$$

Remark. It was pointed out to me by Ilya Kapovich that the results of [KW03] imply that the free subgroups provided by Theorem 1 are quasiconvex in $\pi_1(M_i)$.

The principal ingredient of the proof of the theorem is the following result of an independent interest. Let $\mathrm{sysg}(M)$ denote the minimal genus of a surface subgroup of $\pi_1(M)$, which we call the *systolic genus* of M.

Theorem 2. *Let M be a closed hyperbolic 3-manifold. For any $\epsilon > 0$, assuming that the systole $\mathrm{sys}_1(M)$ is sufficiently large, we have*

$$\mathrm{sysg}(M) \geq e^{(\frac{1}{2}-\epsilon)\mathrm{sys}_1(M)}.$$

In particular, given a sequence of closed hyperbolic 3-manifolds M with $\mathrm{sys}_1(M) \to \infty$, we have

$$\log \mathrm{sysg}(M) \gtrsim \frac{1}{2}\mathrm{sys}_1(M).$$

The proof of Theorem 2 is based on Thurston's inequality bounding the area of a π_1-injective surface in a hyperbolic 3-manifold through its genus and an important Gromov's systolic inequality for surfaces of high genus, in which we use a numerical value of the constant obtained by Katz–Sabourau. The application of these ingredients is supported by the results from the theory of minimal surfaces of Schoen–Yau and Sacks–Uhlenbeck. We refer to [Be12, Section 2] for the details of the proof and precise references.

In order to relate Theorem 2 and $\mathcal{N}_{fr}(M)$, we recall the following result of Baumslag and Shalen [BaSh89]:

Theorem 3. *Let M be an irreducible, closed orientable 3-manifold, and let k be a positive integer. Suppose that $\pi_1(M)$ has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$, and that $\beta_1(M) > k$. Then $\pi_1(M)$ is k-free.*

By the work of Xue [Xue92], if M is an arithmetic hyperbolic 3-manifold whose group is defined by a quadratic form and $M_i \to M$ is a sequence of its congruence covers, then $\log \beta_1(M_i) \gtrsim \frac{1}{3} \log \operatorname{vol}(M_i)$. The analogues results about Betti numbers are known also for some other families of arithmetic hyperbolic 3-manifolds and are conjectured to be true in general (see [Be12] for more details). In order to finish the proof of Theorem 1, it remains to recall that the systole of the congruence covers of an arithmetic hyperbolic 3-manifold grows at least as fast as $\frac{2}{3} \log \operatorname{vol}(M_i)$ [KSV07] and to combine all these facts together with Theorem 2.

Most of the above mentioned results hold for higher dimensional hyperbolic manifolds and their congruence covers, and we expect Theorem 1 to be true in higher dimensions as well. The principal problem in obtaining such a generalisation is with Theorem 3 whose proof in [BaSh89] is essentially 3-dimensional.

References

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