KAZHDAN–LUSZTIG CELLS IN INFINITE COXETER GROUPS

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Abstract. We introduce two conjectures which can be used to describe Kazhdan-Lusztig cells in arbitrary infinite Coxeter groups.

1. Introduction

Groups defined by presentations of the form $\langle s_1, \dots, s_n \mid s_i^2 = 1,\ (s_i s_j)^{m_{i,j}} = 1\ (i, j = 1, \dots, n)\rangle$ are called *Coxeter groups*. The exponents $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ form the Coxeter matrix, which characterizes the group up to isomorphism. The Coxeter groups that are most important for applications are the Weyl groups and affine Weyl groups. For example, the symmetric group S_n is isomorphic to the Coxeter group with presentation $\langle s_1, \dots s_n \mid s_i^2 = 1\ (i = 1, \dots, n),\ (s_i s_{i+1})^3 = 1\ (i = 1, \dots, n-1)\rangle$, and is also known as the Weyl group of type A_{n-1}. For more information about Coxeter groups we refer to the book [H].

The notion of *cells* was introduced by Kazhdan and Lusztig [KL] to study representations of Coxeter groups and their Hecke algebras. Later it was realized that cells arise in many different branches of mathematics and have many interesting properties. Some examples and references for the related results can be found in [G].

We are interested in the combinatorial structure of the cells in infinite Coxeter groups. Examples of such groups include the affine Weyl groups, as well as Weyl groups of hyperbolic Kac–Moody algebras. In contrast with the affine case, there are very few results on cells in hyperbolic Coxeter groups available so far. We refer to [Bed] and [Bel] for some work in this direction. Based on numerous computational experiments we introduce two conjectures that describe the structure of the cells in infinite Coxeter groups. Our conjectures use *combinatorial rigidity* for elements in an infinite Coxeter group (§4). We expect that this notion may be of independent interest. We were able to check the conjectures for affine groups of small rank (see [BG2] and §5), and to prove them in some special cases (see [BG1] and §4). The latter include, in particular, the right-angled Coxeter groups previously studied in [Bel]. The proof of the conjectures in their general form remains an open problem.

2. Visualization of cells

Let W be a Coxeter group with a fixed system of generators S. Consider a real vector space V of dimension $|S|$ with a basis $\{\alpha_s \mid s \in S\}$. We define a symmetric bilinear form on V by

$$B(\alpha_s, \alpha_t) = -\cos(\pi/m(s,t)),\ \ s, t \in S.$$

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Now for every $s \in S$ we can define a linear map $\sigma_s \colon V \to V$ by

$$\sigma_s(\lambda) = \lambda - 2B(\alpha_s, \lambda)\alpha_s.$$

This map sends α_s to $-\alpha_s$ and fixes the hyperplane H_s orthogonal to α_s with respect to B in V. Therefore, σ_s is a reflection of the space V. One can show that the map $s \to \sigma_s$ extends to a faithful linear representation of the group W in $GL(V)$, called the *standard geometric realization* of the group.

Next let us introduce the *Tits cone* $\mathcal{C} \subset V$ of W. Every hyperplane H_s divides V into two halfspaces. Let H_s^+ denote the closed halfspace on which the element α_s^* dual to α_s is nonnegative. The intersection of these halfspaces $\Sigma_0 = \cap H_s^+$ for $s \in S$ is a closed simplicial cone in V. The closure of the union of all W-translates of Σ_0 is again a cone in V. This cone $\mathcal{C}$ is called the Tits cone. It is known that $\mathcal{C} = V$ if and only if W is finite. For infinite groups W the Tits cone is significantly smaller than the whole space and hence is more convenient for the geometric realization of the group.

Under some additional assumptions we can define the action of W on a section of the Tits cone. In particular, this way we can describe the action of affine or hyperbolic Coxeter groups of rank 3 on a Euclidean or Lobachevsky plane, respectively. For example, consider the affine group W of type $\widetilde{\mathrm{A}}_2$. This group is generated by three involutions s_1, s_2, s_3 satisfying the relations $(s_i s_j)^3 = 1$ for all $i \neq j$. The space V is isomorphic to $\mathbb{R}^3$, and via this the Tits cone $\mathcal{C}$ can be identified with the upper halfspace $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. It is easy to check that the action of W preserves the affine plane $M := \{z = 1\}$, and that the images of Σ_0 intersect M in equilateral triangles (Figure 1). This construction implies that the group W of type $\widetilde{\mathrm{A}}_2$ can be realized as the discrete subgroup of the affine isometries of a plane generated by reflections in the sides of an equilateral triangle.

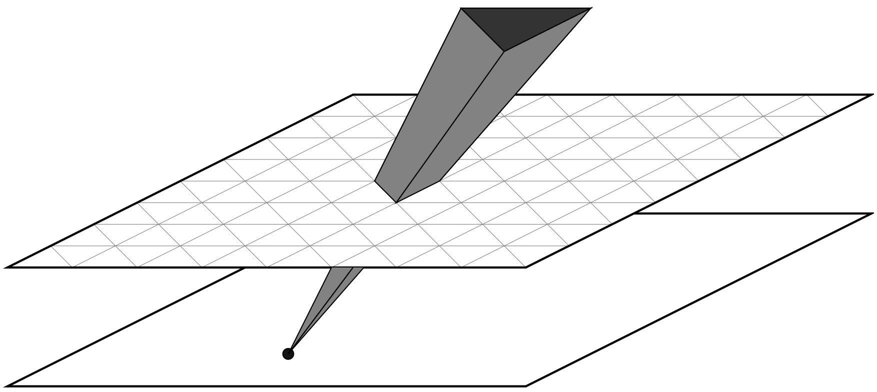


FIGURE 1. Slicing the Tits cone

A generalization of this construction leads to *triangle groups*. Let Δ be a triangle with angles π/p, π/q, π/r, where $p, q, r \in \mathbb{N} \cup \{\infty\}$. The triangle Δ lives on a sphere, or in an affine or hyperbolic plane, depending on whether $1/p + 1/q + 1/r$ is > 1, $= 1$, or < 1. The group W_{pqr} of isometries of the corresponding space generated by reflections in the sides of Δ is respectively a finite, affine, or hyperbolic Coxeter group. For example W_{333} is the affine group of type $\widetilde{\mathrm{A}}_2$. In Figure 2 we show the tessellation of the hyperbolic plane corresponding to W_{237} (the Hurwitz group). The coloring of the triangles indicates the partition of W into cells, which we will define in the next section.

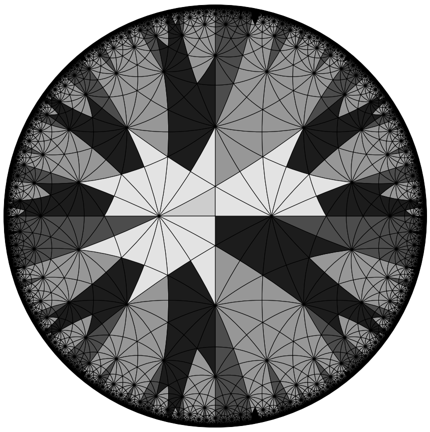


FIGURE 2. W_{237}

3. Main definitions

Consider a Coxeter group W with a system of generators S. Any element $w \in W$ can be written as a product, or *word*, in the generators: $w = s_1 \dots s_N$, $s_i \in S$. Such an expression is called *reduced* if we cannot use the relations in W to produce a shorter expression for w. An element can have different reduced expressions but it is not hard to check that all of them have the same length. Therefore, we can define the *length function* $l\colon W \to \mathbb{N} \cup \{0\}$, which assigns to an element $w \in W$ the length of a reduced expression with respect to the generators S (see [H, Ch. 1.6] for more details). Another important notion which can be defined using the reduced expressions is the *partial order* $\leq$ *of Chevalley–Bruhat*. Let $s_1 \dots s_N$ be a word in the generators. We define a *subexpression* as to be any (possibly empty) product of the form $s_{i_1} \dots s_{i_M}$, where $1 \leq i_1 \leq \dots \leq i_M \leq N$. We say that $y \leq w$ if an expression for y appears as a subexpression of a reduced expression for w. It can be shown that the relation $\leq$ is a partial order on the group W (see [H, Ch. 5.9]).

Let $\mathcal{H}$ denote the Hecke algebra of W over the ring $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in $q^{1/2}$. This algebra is a free $\mathcal{A}$-module with a basis T_w, $w \in W$ and with multiplication defined by $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$, and $T_s^2 = q + (q-1)T_s$ for $s \in S$. Together with the basis $(T_w)_{w\in W}$, we can define in $\mathcal{H}$ another basis $(C_w)_{w\in W}$. This new basis, introduced by Kazhdan and Lusztig in [KL], has a number of important properties and has proven to be very convenient for describing the representations of W and $\mathcal{H}$. The elements C_w can be expressed in terms of T_w by the formulae

$$C_w = \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{l(w)/2 - l(y)} P_{y,w}(q^{-1}) T_y,$$

where the $P_{y,w}(t) \in \mathbb{Z}[t]$ are the *Kazhdan–Lusztig polynomials*. The polynomials $P_{y,w}(t)$ are nonzero exactly when $y, w \in W$ satisfy $y \leq w$, equal 1 when $y = w$, and otherwise have degree $\deg(P_{y,w})$ at most $d(y,w) := (l(w) - l(y) - 1)/2$. If $\deg(P_{y,w}) = d(y,w)$, we denote the leading coefficient by $\mu(y,w) = \mu(w,y)$, and in all other cases (including when y and w are not comparable in the partial order) we put $\mu(y,w) = \mu(w,y) = 0$. We indicate that $\mu(y,w) \neq 0$ by y—w.

Using the polynomials $P_{y,w}$ we can define the partial orders $\leq_L$, $\leq_R$, $\leq_{LR}$ on W. First, for $w \in W$ we define the *left* and *right descent sets*:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\}, \quad \mathcal{R}(w) = \{s \in S \mid ws < w\}.$$

Next, we say that $y \leq_L w$ if there exists a sequence $y = y_0, y_1, \ldots, y_n = w$ in W such that y_i—y_{i+1} and $\mathcal{L}(y_i) \not\subset \mathcal{L}(y_{i+1})$ for all $0 \leq i < n$. The relation $\leq_R$ can be defined using $\leq_L$: we put $y \leq_R w$ if $y^{-1} \leq_L w^{-1}$. Finally, $y \leq_{LR} w$ means that there exists a sequence $y = y_0, y_1, \ldots, y_n = w$ such that for all $i < n$, we have either $y_i \leq_L y_{i+1}$ or $y_i \leq_R y_{i+1}$. It is easy to check that $\leq_L$, $\leq_R$ and $\leq_{LR}$ are partial orders on W; we denote the corresponding equivalence relations by $\sim_L$, $\sim_R$ and $\sim_{LR}$. The equivalence classes of $\sim_L$ (respectively, $\sim_R$, $\sim_{LR}$) are called the *left cells* (resp. *right cells*, *two-sided cells*) of W. It follows from the definitions that the left and right cells have very similar properties, thus we will mostly deal with only one of these two types of cells.

Consider for example Figure 2, which depicts the cells of W_{237}. There are five 2-sided cells, corresponding to the five shades of gray in the figure. The left cells can also be seen: they are the connected unions of triangles of a given color. Note that this group apparently has infinitely many left cells.

Now consider the multiplication of the C-basis element in $\mathcal{H}$. We can write

$$C_x C_y = \sum_z h_{x,y,z} C_z, \; h_{x,y,z} \in \mathcal{A}.$$

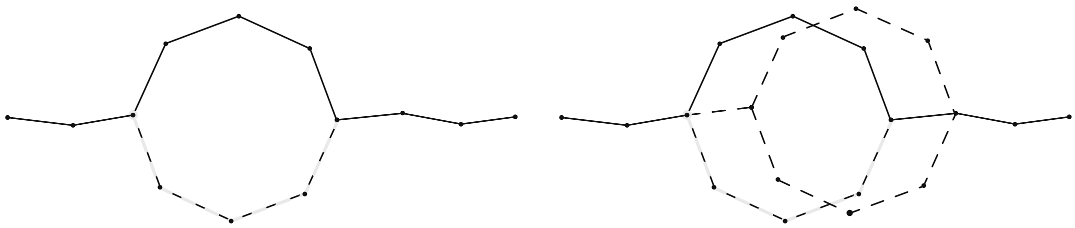
Let $a(z)$ be the smallest integer such that $q^{a(z)/2} h_{x,y,z} \in \mathcal{A}^+$ for all $x, y \in W$, where $\mathcal{A}^+ = \mathbb{Z}[q^{1/2}]$. Denote by $\mathcal{D}_i$ the set $\{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}$, where $\delta(z)$ is the degree of the polynomial $P_{e,z}$, $l(z)$ is the length function on W, and $a(z)$ is defined as above. The set $\mathcal{D} = \mathcal{D}_0$ consists of *distinguished involutions* of W introduced by Lusztig in [L2, 1.3].

Lusztig proved that in affine groups there is a bijection between the set $\mathcal{D}$ and the left (or right) cells of W [L2]. This deep result has many important corollaries and applications. One of the main goals of our work is to make this correspondence between distinguished involutions and cells explicit, so that one can describe the structure of the cells using the distinguished involutions of the group. Another goal is to find an algorithm that produces distinguished involutions of a given group. In the next section we will formulate two conjectures that answer these questions and present some results to support the conjectures.

4. Conjectures and results

We will need some more notations and definitions. Given $w \in W$, we write $w = x.y$ if $w = xy$ and $l(w) = l(x) + l(y)$. Denote by $Z(w)$ the set of all $v \in W$ such that $w = x.v.y$ for some $x, y \in W$ and $v \in W_I$ for some $I \subset S$ with W_I finite. (We recall that for a subset $I \subset S$, W_I denotes the standard parabolic subgroup of W generated by $s \in I$.) We call $v \in Z(w)$ *maximal in* w if it is not a proper subword of any other $v' \in Z(w)$ such that $w = x'.v'.y'$ with $x' \leq x$ and $y' \leq y$. Let $Z = Z(W)$ be the union of $Z(w)$ over all $w \in W$, $\mathcal{D}_f := \mathcal{D} \cap Z$ be the set of distinguished involutions of the finite standard parabolic subgroups of W and $\mathcal{D}_f^\bullet := \mathcal{D}_f \smallsetminus (S \cup \{1\})$. Note that each of the sets $Z(W)$, $\mathcal{D}_f$ and $\mathcal{D}_f^\bullet$ is finite.

We call $w = x.v.y$ *rigid at* v if (i) $v \in \mathcal{D}_f$, (ii) v is maximal in w, and (iii) for every reduced expression $w = x'.v'.y'$ with $a(v') \geq a(v)$, we have $l(x) = l(x')$ and $l(y) = l(y')$. This notion of *combinatorial rigidity* plays an important role in our considerations. Figure 3 helps to understand its meaning using the Cayley graph of the group W. Maximal distinguished involutions of the finite parabolic subgroups correspond to the "long cycles" in the graph. Combinatorial rigidity means that such a cycle can not be shifted along the presentation of w in any direction. For example, in the triangle group W_{333} of type $\widetilde{\mathrm{A}}_2$, the element $w = s_3s_1s_2s_1s_3$ is rigid at $v = s_1s_2s_1$, but $w' = s_2s_3s_1s_2s_1s_3s_2$ is not rigid at v.



(a) (b)

FIGURE 3. Rigid(a) and non-rigid(b) expressions in the Cayley graph

Our conjectures can be formulated as follows:

Conjecture 1. *("distinguished involutions") Let $v = x.v_1.x^{-1} \in \mathcal{D}$ with $v_1 \in \mathcal{D}_f^\bullet$ and $a(v) = a(v_1)$, and let $v' = s.v.s$ with $s \in S$. Then if sxv_1 is rigid at v_1, we have $v' \in \mathcal{D}$.*

Conjecture 2. *("basic equivalences") Let $w = y.v_0$ with v_0 maximal in w.*

(a) *Let $u = x.v_1.x^{-1} \in \mathcal{D}$ satisfies $a(u) \le a(v_0)$ and $w' = wu$ is reduced and has $a(w') = a(w)$. Then there exists v_{01} such that $v_0 = v_0'.v_{01}$, $v_{01}'xv_1$ is rigid at v_1 for every v_{01}' such that $v_0 = v_0''.v_{01}'$ and $l(v_{01}') = l(v_{01})$, the right descent set $\mathcal{R}(w'v_{01}^{-1}) \subsetneq \mathcal{R}(w)$, and $\mu(w, w'v_{01}^{-1}) \ne 0$, which implies $w \sim_R w'v_{01}^{-1} \sim_R w'$.*
(b) *Let $w'' = w.v_1$ with $v_1 \in \mathcal{D}_f$ not maximal in w'' and $a(w'') = a(v_0)$. Then we can write $w = y.v_{01}.v_{02}.v_{03}$ so that $v_{03}.v_1$ is maximal in w'', $\mathcal{R}(w''v_{02}^{-1}) \ne \mathcal{R}(w)$, and $\mu(w, w''v_{02}^{-1}) \ne 0$. So again $w \sim_R w''v_{02}^{-1} \sim_R w''$.*

Conjecture 1 can be used to inductively construct distinguished involutions in an infinite Coxeter group W starting from the involutions of its finite standard parabolic subgroups. Conjecture 2, in turn, allows one to obtain equivalences in the group using its distinguished involutions. Let us note that the usual method for obtaining results of this kind is based on computing Kazhdan–Lusztig polynomials. This requires many computations and, moreover, does not give any a priori information about the elements that are distinguished involutions or satisfy the cell equivalences in the group. For infinite Coxeter groups in which an exhaustive search is not possible this latter disadvantage becomes critical. Our Conjecture 2 does not necessarily give *all* equivalences in the group, but still one can expect that the equivalences provided by the conjecture suffice for describing the cells. More precisely, we have the following theorem.

Theorem 1. [BG1] *If an infinite Coxeter group satisfies Conjectures 1 and 2, and also two conjectures of Lusztig, then*

(1) *The set $\mathcal{D}$ of distinguished involutions consists of the union of $v \in \mathcal{D}_f$ and the elements of W obtained from them using Conjecture 1.*
(2) *The relations described in Conjecture 2 determine the partition of W into right cells.*
(3) *The relations described in Conjecture 2 together with its $\sim_L$-analogue determine the partition of W into two-sided cells.*

The conjectures of Lusztig to which we refer in the theorem are so-called "positivity conjecture" and a conjecture about a combinatorial description of the function $a(z)$. The positivity conjecture is now proved for a wide class of infinite Coxeter groups that includes affine Weyl groups. On the other hand, there have been recently found examples of Coxeter groups for which the conjecture about the function $a(z)$ is false. We expect that even for these groups the conclusion of Theorem 1 is true but it would require a different argument. We refer to [BG1] for more details and the proof of the theorem.

We were able to prove Conjectures 1 and 2 under certain additional assumptions. The results are given in the following two theorems.

Theorem 2. [BG1] *Let $v = x.v_1.x^{-1} \in \mathcal{D}$ with $v_1 \in \mathcal{D}_f^\bullet$, $a(v) = a(vs)$, and $\mathcal{L}(vs) \smallsetminus \mathcal{R}(vs) \neq \emptyset$. Then if $v' = s.v.s$ is rigid at v_1, we have $v' \in \mathcal{D}$.*

Theorem 3. [BG1] *Let $w = x.v_0 = t_n \dots t_1.s_l \dots s_1$ with $t_i, s_i \in S$, $v_0 = s_l \dots s_1 \in \mathcal{D}_f$ is the longest element of a standard finite parabolic subgroup of W which is maximal in w and $a(w) = a(v_0)$; $u = y.u_0.y^{-1} \in \mathcal{D}$ with $u_0 \in \mathcal{D}_f$ such that $a(u) = a(u_0) = l$; and $w' = w.u.v_{01}$ with $v_{01} = s_1 \dots s_{l-1}$ has $a(w') = a(w)$ and $\mathcal{R}(w') \subsetneq \mathcal{R}(w)$.*

Assume that

(1) *For any $v_j = t_j \dots t_1 v_0 t_1 \dots t_j$, $j = 0, \dots, n-1$ and $t = t_{j+1}$ or $t = t_{j-1}$ if $t_{j-1} \notin \mathcal{R}(v_j)$, we have $a(v_j t) = a(v_j)$, $\mathcal{L}(v_j t) \smallsetminus \mathcal{R}(v_j t) \neq \emptyset$ and $t v_j t$ is rigid at v_0.*
(2) *For any $u_j = s_{j-1} \dots s_1 u s_1 \dots s_{j-1}$, $j = 1, \dots, l-1$ with $u_1 = u$, we have $a(u_j s_j) = a(u_j)$, $\mathcal{L}(u_j s_j) \smallsetminus \mathcal{R}(u_j s_j) \neq \emptyset$ and $s_j u_j s_j$ is rigid at u_0;*

Then $\mu(w, w') \neq 0$ and $w \sim_R w'$.

The additional assumption $\mathcal{L}(vs) \smallsetminus \mathcal{R}(vs) \neq \emptyset$ in Theorem 2 may seem minor, but unfortunately this is not the case. In particular, conditions (1) and (2) in Theorem 3 appear as a consequence of this assumption. The proof of the theorems 2 and 3 in [BG1] essentially uses the results from two unpublished letters of Springer and Lusztig [LS]. A possible approach to the proof of our conjectures in general requires developing further the ideas of this correspondence.

Although the conditions of Theorems 2 and 3 are not always met for all W, the theorems can still be used to produce interesting results. We will give some examples in the next section, other applications of the theorems are considered in [BG1].

5. Cells in affine groups of rank 3

Affine Weyl groups of rank 3 have type $\widetilde{\mathrm{A}}_2$, $\widetilde{\mathrm{B}}_2$ ($= \widetilde{\mathrm{C}}_2$) or $\widetilde{\mathrm{G}}_2$ (see [H, Chapter 4]). The cells in these groups were first described by Lusztig in [L1]. In this section we will show how the same results can be relatively easily obtained using conjectures from §4.

Type $\widetilde{\mathrm{A}}_2$: The group W is generated by involutions s_1, s_2, s_3 with relations $(s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = 1$. We have

$$\mathcal{D}_f = \{1, s_1, s_2, s_3, s_1 s_3 s_1, s_3 s_2 s_3, s_2 s_1 s_2\},\ \mathcal{D}_f^\bullet = \{s_1 s_3 s_1, s_3 s_2 s_3, s_2 s_1 s_2\}.$$

Applying Conjecture 1 with $v = v_1 = s_1 s_3 s_1$, we get $v' = s_2 v s_2 \in \mathcal{D}$. Note that after this the inductive procedure terminates as the elements $s_1 s_2 v$ and $s_3 s_2 v$ which would come out on the next step are both non-rigid at v. We can apply the same procedure to the other two involutions from $\mathcal{D}_f^\bullet$. As a result we get

$$\mathcal{D} = \{1, s_1, s_2, s_3, s_1s_3s_1, s_3s_2s_3, s_2s_1s_2, s_2s_1s_3s_1s_2, s_1s_3s_2s_3s_1, s_3s_2s_1s_2s_3\}.$$

Therefore, the group W of type $\widetilde{\mathrm{A}}_2$ has 10 left (right) cells. Using Conjecture 2 and the geometric realization of the group it is easy to show that the partition of W into cells is the one shown on Figure 4(a), where two-sided cells correspond to the regions of the same color and left cells correspond to the connected components of the two-sided cells. This coincides with the result of [L1]. Note that in order to produce the cells we only need Theorems 2 and 3, and thus our results for this case are unconditional.

Type $\widetilde{\mathrm{B}}_2$: The group W is generated by involutions s_1, s_2, s_3 with $(s_1s_2)^2 = (s_2s_3)^4 = (s_1s_3)^4 = 1$.

$$\mathcal{D}_f^\bullet = \{s_1s_2, (s_2s_3)^2, (s_1s_3)^2\}.$$

Conjecture 1 gives

$$\begin{aligned}\mathcal{D} = \{&1, s_1, s_2, s_3, s_1s_2, s_3s_1s_2s_3, s_1s_3s_1s_2s_3s_1, s_2s_3s_1s_2s_3s_2,\\ &s_2s_3s_2s_3, s_1s_2s_3s_2s_3s_1, s_3s_1s_2s_3s_2s_3s_1s_3, s_2s_3s_1s_2s_3s_2s_3s_1s_3s_2,\\ &s_1s_3s_1s_3, s_2s_1s_3s_1s_3s_2, s_3s_2s_1s_3s_1s_3s_2s_3, s_1s_3s_2s_1s_3s_1s_3s_2s_3s_1\}.\end{aligned}$$

The partition of W into cells, which we get using Conjecture 2, is shown on Figure 4(b). One can quite easily obtain this partition by following the cycles which correspond to the distinguished involutions on the tessellation of the plane. The result is again in agreement with [L1]. Note that in this case one can check that the assumptions of Theorem 2 hold, but not the assumptions of Theorem 3. Thus we can compute the distinguished involutions, but we cannot prove that our relations suffice to generate the cells, and therefore cannot be sure that we actually have all the distinguished involutions without using Lusztig's computations in [L1] or referring to our conjectures.

Type $\widetilde{\mathrm{G}}_2$: The group W is generated by involutions s_1, s_2, s_3 with $(s_1s_2)^2 = (s_3s_1)^3 = (s_2s_3)^6 = 1$.

$$\mathcal{D}_f^\bullet = \{s_1s_2, s_3s_1s_3, (s_2s_3)^3\}.$$

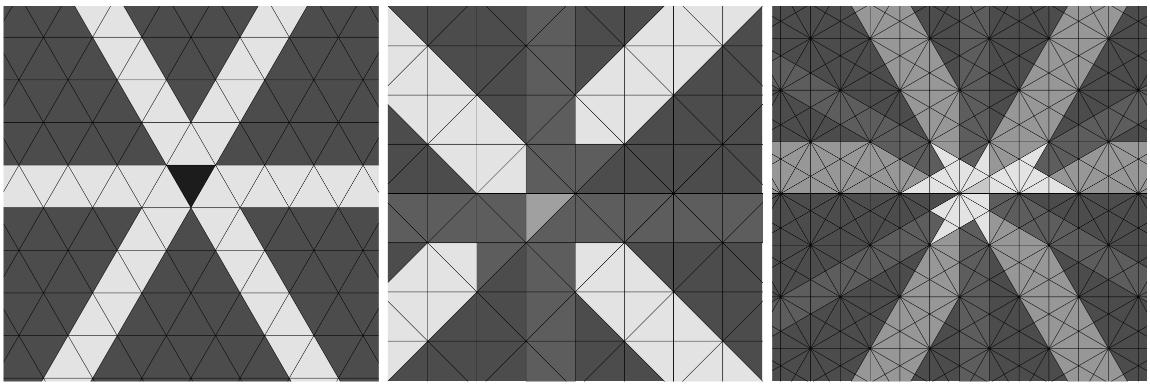
The application of Conjecture 1 in this case already requires some effort because of a large number of possible variants. In order to generate the list of distinguished involutions we used a computer. Our algorithms and their application to other affine Weyl groups are described in [BG2]. As a result of these computations, we obtain

$$\begin{aligned}\mathcal{D} = \{&1, s_1, s_2, s_3, s_1s_2,\\
&s_3s_1s_2s_3, s_2s_3s_1s_2s_3s_2, s_3s_2s_3s_1s_2s_3s_2s_3, s_1s_3s_2s_3s_1s_2s_3s_2s_3s_1, s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2,\\
&s_3s_1s_3, s_2s_3s_1s_3s_2, s_3s_2s_3s_1s_3s_2s_3, s_2s_3s_2s_3s_1s_3s_2s_3s_2,\\
&s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_3, s_1s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_3s_1,\\
&s_2s_3s_2s_3s_2s_3, s_1s_2s_3s_2s_3s_2s_3s_1, s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3, s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2,\\
&s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3, s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2,\\
&s_1s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_1, s_2s_1s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_1s_2,\\
&s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_1s_3,\\
&s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_1s_3s_2,\\
&s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_1s_3s_2s_3,\\
&s_1s_3s_2s_3s_1s_2s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_2s_1s_3s_2s_3s_1\}.\end{aligned}$$

Therefore, the group W of type $\widetilde{\mathrm{G}}_2$ has 28 left (right) cells. The interested reader can check that Conjecture 2 allows us to obtain the partition of W into cells which is shown on Figure 4(c). Note that for this case the conditions of neither Theorem 2 nor 3 are satisfied. For instance, we have

$$\mathcal{L}(s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_1) \setminus \mathcal{R}(s_3s_2s_3s_1s_2s_3s_2s_3s_2s_3s_1s_3s_2s_3s_1) = \emptyset.$$

Thus our results here rely on unproved instances of the conjectures, but nevertheless the results agree with [L1].



(a) (b) (c)

FIGURE 4. Cells in $\widetilde{\mathrm{A}}_2$, $\widetilde{\mathrm{B}}_2$ and $\widetilde{\mathrm{G}}_2$

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