Counting Lattices

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Let H be a non-compact simple Lie group endowed with a fixed Haar measure μ. Let $\mathrm{L}_H(x)$ (resp. $\mathrm{AL}_H(x)$) denote the number of conjugacy classes of lattices (resp. arithmetic lattices) in H of covolume at most x.

A classical theorem of Wang [W] asserts that if H is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{L}_H(x)$ is finite for every x. This is also true for $\mathrm{AL}_H(x)$ even for $H = \mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ by a result of Borel [Bo].

Recent years there has been a growing interest in the asymptotic behavior of these functions.

In [BGLM] the rate of growth of torsion-free lattices was determined for $H = \mathrm{SO}(n,1)$, $n \geq 4$; it is super-exponential. The lower bound there is already obtained by considering a suitable single lattice in $\mathrm{SO}(n,1)$ and its finite index subgroups. The upper bound is proved by geometric methods.

In [BGLS] we give a very precise super-exponential estimate for $\mathrm{AL}_H(x)$ for $H = \mathrm{SL}_2(\mathbb{R})$. Our main result states that $\lim_{x\to\infty} \frac{\log \mathrm{AL}_H(x)}{x \log x} = \frac{1}{2\pi}$. Here again the full rate of growth is already obtained by considering the finite index subgroups of a single lattice — the main challenge is in proving the upper bound.

In [GLP] and [LN] (see also [GLNP]) precise asymptotic estimates were given for the growth rate of the number of congruence subgroups in a fixed lattice Λ in H. (Some of the results there are conditional on the GRH). That rate of growth turns out to depend only on H and not on Λ.

All this suggested that the rate of growth of the finite index subgroups within one lattice is the main contribution to $\mathrm{L}_H(x)$. This led to the following *conjecture* (see e.g. [GLNP]):

Let H be a non-compact simple Lie group of real rank at least 2. Then

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H(x)}{(\log x)^2/\log\log x} = \gamma(H), \quad \textit{with} \;\; \gamma(H) = \frac{(\sqrt{h(h+2)} - h)^2}{4h^2},$$

where h is the Coxeter number of the (absolute) root system corresponding to H (i.e. the root system of the split form of H).

In [B] it is shown that the growth rate of the maximal arithmetic lattices in H is very small (conjecturally polynomial, and indeed a polynomial bound is given there for the maximal non-uniform lattices and a slightly weaker bound of the form $x^{(\log x)^\epsilon}$ is proved for all maximal lattices). This gave a further support to the conjecture.

In [BL2] we show that the conjecture is essentially true for non-uniform lattices but in [BL1] we prove, somewhat surprisingly, that it is false in general. In fact, we discover here a new phenomenon: the main contribution to the growth of uniform lattices in H does not come from subgroups of a single lattice. As it will be explained below, it comes from a "diagonal counting" when we run through different arithmetic groups Γ_i defined over number fields k_i of different degrees d_i, and for each Γ_i we count some of its subgroups. The difference between the uniform and non-uniform cases relies on the fact that all non-uniform lattices in H are defined over number fields of a bounded degree over $\mathbb{Q}$. On the other hand, uniform lattices may come from number fields k_i of arbitrarily large degrees, i.e., $d_i \to \infty$.

We now briefly describe the line of the argument. If Γ is an arithmetic lattice, there exists a number field k with ring of integers $\mathcal{O}$ and the set of archimedean valuations V_∞, an absolutely simple, simply connected k-group G and an epimorphism $\phi : G = \prod_{v \in V_\infty} \mathrm{G}(k_v)^o \to H$, such that $\mathrm{K}er(\phi)$ is compact and $\phi(\mathrm{G}(\mathcal{O}))$ is commensurable with Γ. G. Prasad [P] gave an explicit formula for the covolume of such $\phi(\mathrm{G}(\mathcal{O}))$ in H. The analysis of this formula and also the growth of the low-index congruence subgroups of $\phi(\mathrm{G}(\mathcal{O}))$ shows that we can expect fast subgroup growth if we consider groups over fields of growing degree with relatively slow growing discriminant $\mathcal{D}_k$. More precisely, we can combine this two entities together into the so-called root-discriminant $rd_k = \mathcal{D}_k^{1/\deg k}$ and then look for a sequence of number fields k_i with degrees growing to ∞ but with bounded rd_{k_i}. In a seminal work Golod and Shafarevich [GS] came up with a construction of infinite class field towers. It is such a tower of number fields k_i that we use to define our arithmetic subgroups Γ_i. Galois cohomology methods show the existence of suitable k_i-algebraic groups G_i which give rise to arithmetic lattices $\Gamma_i = \mathrm{G}_i(\mathcal{O}_i)$ in H whose covolume is bounded exponentially in $d_i = \deg k_i$. We then present $c^{d_i^2}$ congruence subgroups of Γ_i whose covolume is still bounded exponentially in d_i. Using the theory of Bruhat-Tits buildings we can show that sufficiently many of such congruence subgroups are not conjugate to each other in H. This completes the proof of the lower bound $\log \mathrm{L}_H(x) \geq a(\log x)^2$ for some positive constant $a = a(H)$ at least for most real simple Lie groups H. The remaining cases require further consideration: for example, if H is a complex Lie group, the fields k_i should be replaced by suitable extensions obtained via the help of the theory of Pisot numbers. These fields do not form a class field tower any more but still have bounded root discriminant.

The proof of the upper bound $\log \mathrm{L}_H(x) \leq b(\log x)^2$ for groups H of real rank at least 2 which satisfy Serre's congruence subgroup conjecture in [BL1] presents a new type of difficulty: this time we need to obtain a uniform upper bound on growth which does not depend on the degrees of the defining fields. (This is what makes the growth rate $x^{\log x}$ instead of $x^{\log x / \log \log x}$.) The new bound requires some new "subgroup growth" methods which we develop in [BL1]. A key ingredient of the proof is an important theorem of Babai, Cameron and Pálfy (see [LS, Theorem 4, p. 339]) which bounds the size of permutation groups with restricted Jordan-Holder components. We are taking advantage of the fact that this restriction applies uniformly for the profinite completions of all the lattices in a given group H.

On the other hand, the result of [BL2] shows that if one restricts attention only to non-uniform lattices then the original conjecture is true for most higher rank simple groups H (and conjecturally for all). Thus, let us assume that if G is a split form of H, then the center of the simply connected cover of G is a 2-group, and that H is not a triality. This is the case for most H's. In fact, it says that H is not of type E_6 or D_4, and if it is of type A_n, then n is of the form $n = 2^\alpha - 1$ for some $\alpha \in \mathbb{N}$. For such H we can show that $\lim_{x\to\infty} \frac{\log \mathrm{L}_H^{nu}(x)}{(\log x)^2 / \log \log x} = \gamma(H)$, where $\gamma(H)$ is defined as above and $\mathrm{L}_H^{nu}(x)$ denotes the number of conjugacy classes of non-uniform lattices in H of covolume at most x.

The proof of of this result uses Gauss's Theorem which gives a bound for the 2-rank of the class groups of quadratic extensions. In order to be able to extend the result to all simple groups H we would need similar bounds for l-ranks for $l > 2$. In fact, we show in [BL2] that it is essentially equivalent to such bounds.

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