Subspace stabilisers in hyperbolic lattices

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- For $n = 3$ this answers a question of A. Reid and C. McMullen.
- Arithmetic hyperbolic 3-orbifolds not of the *simplest type* and 2-orbifolds are excluded.
- The result tells very little about the nature of totally geodesic subspaces.

FC-subspaces

Definition. A totally geodesic subspace N of a hyperbolic orbifold $M = \mathbf{H}^n/\Gamma$ is called a *finite centraliser subspace* (or an *fc-subspace*) if there exists a finite subgroup $F < \mathrm{Comm}(\Gamma)$ such that $H = \mathrm{Fix}(F)$ is a subspace of $\mathbf{H}^n$ and $N = H/\,\mathrm{Stab}_\Gamma(H)$.

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Here $\mathrm{Fix}(F) = \{x \in \mathbf{H}^n \,|\, gx = x,\ \forall g \in F\}$, and
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Important Property: *An fc-subspace of dimension $\geqslant 2$ of a finite volume hyperbolic orbifold is a finite volume hyperbolic orbifold.*

A dichotomy

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Let M be a finite volume hyperbolic n-orbifold, $n \geqslant 2$.

- *If M is arithmetic, then all the totally geodesic subspaces of codimension at most $\frac{n+1}{2}$ are fc-subspaces;*
- *If M is non-arithmetic, then the number of its fc-subspaces is bounded by $c\,\mathrm{Vol}(M)$, $c = \mathrm{const}(n)$.*

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Corollary 1. If M is an arithmetic hyperbolic 3-orbifold, then *all* its totally geodesic subspaces are fc.

Corollary 2. A finite area hyperbolic surface is arithmetic if and only if all of its infinitely many closed geodesics are fc-subspaces.

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Proof of Theorem 2 is based on:

- Borel's density theorem;
- *A construction of involutions for each of the three types of arithmetic lattices;*
- Margulis superrigidity theorem.

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Let M be a quasi-arithmetic hyperbolic orbifold with adjoint trace field k, and $N \subset M$ be a finite-volume totally geodesic suborbifold of dimension $m \geqslant 2$ with adjoint trace field K. Then N is hyperbolic and quasi-arithmetic, and $k \subseteq K$. If M is arithmetic, then N is arithmetic as well.

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Prop. 1. *Let $N = \mathbf{H}^m/\Lambda$ be a subform space of an arithmetic orbifold $M = \mathbf{H}^n/\Gamma$. Then N is an fc–subspace associated to a single involution in the commensurator of Γ.*

Prop. 2. *Let M be a type I (resp. type II) arithmetic hyperbolic orbifold, and $N \subset M$ a subform space in M of dimension $\geqslant 2$. Then N is a type I (resp. type II) arithmetic hyperbolic orbifold.*

Main technical results

Theorem 5. [BBKS]
Let $N = \mathbf{H}^m/\Lambda$ be a totally geodesic subspace of an arithmetic hyperbolic orbifold $M = \mathbf{H}^n/\Gamma$. Suppose that N is not a 3-dimensional type III orbifold and that $[K:k] = d \geqslant 1$, where K (resp. k) denotes the adjoint trace field of Λ (resp. Γ). Then there exists a unique minimal subform space $S \subseteq M$ of dimension $(m+1)\cdot d - 1$ such that $N \subseteq S$, and there is no proper subform space of S which contains N.

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Corollary. A totally geodesic immersion $N \subseteq M$ of arithmetic hyperbolic orbifolds is a composition of two geodesic immersions

$$N \subseteq S \subseteq M,$$

where N is a *Weil restriction subspace* of S, and
S is a *subform space* of M.

Example 1

Let $k = \mathbb{Q}$, $V = \mathbb{Q}^{n+1}$ and consider the symmetric bilinear form given in the standard basis by:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus_m \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

where $m = (n-1)/2$.

Let

$$M = \begin{bmatrix} 0 & -1/\sqrt{5} \\ -\sqrt{5} & 0 \end{bmatrix} \oplus_m \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}.$$

Have

$M^t A M = A$, $M^2 = \mathrm{id}$, M corresponds to an involution in $\mathbf{PO}(f)_{\mathbb{Q}}$.

Example 1 (cont.)

The positive eigenspace V^+ relative to the eigenvalue 1 for M has dimension $(n+1)/2$, with orthogonal basis $\mathscr{B}_+$ given by:

$$\mathscr{B}_+ = (e_0 - \sqrt{5}e_1, e_{2i} + (\sqrt{5}-2)e_{2i+1}),\ i = 1, \ldots, m.$$

The negative eigenspace V^- relative to the eigenvalue -1 has the same dimension $(n+1)/2$, with orthogonal basis $\mathscr{B}_-$ given by:

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The restriction g of the form f to V^+ is represented with respect to $\mathscr{B}_+$ by the diagonal matrix with one entry equal to $-2\sqrt{5}$ and all other entries equal to $20-8\sqrt{5}$. Similarly, the restriction h of the form f to V_- is represented with respect to $\mathscr{B}_-$ by the diagonal matrix with one entry equal to $2\sqrt{5}$ and all other entries equal to $20+8\sqrt{5}$.

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Thus g has signature $((n-1)/2, 1)$ and $h = g^\sigma$ is positive definite, so that g is *admissible*.

Example 1 (cont.)

The group

$$\mathrm{Res}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}\mathbf{O}(g)_{\mathbb{R}} = \mathbf{O}(g)_{\mathbb{R}} \times \mathbf{O}(h)_{\mathbb{R}}$$

is realised as the subgroup of $\mathrm{O}(f,\mathbb{R})$ which preserves the decomposition $\mathbb{R}^{n+1} = (V^+ \otimes \mathbb{R}) \oplus (V^- \otimes \mathbb{R})$.

The space $U = \mathbb{H}^n \cap (V^+ \otimes \mathbb{R})$ *projects to an arithmetic finite-volume totally geodesic subspace in* $\mathbb{H}^n/\mathrm{PO}(f,\mathbb{Z})$ *which is a Weil restriction subspace with adjoint trace field* $\mathbb{Q}(\sqrt{5})$ *and ambient group* $\mathbf{PO}(g)$.

Example 2 (type I lattice in a type II lattice).

Let $D' = \left(\frac{-1,3}{\mathbb{Q}}\right)$ a division quaternion algebra and let $K = \mathbb{Q}(\sqrt{3})$.
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Consider the admissible K-form of signature $(2m-1,1)$ given by

$$f(\mathbf{x}) = -\sqrt{3}x_0^2 + x_1^2 + \ldots + x_{2m-1}^2.$$

The form f can be interpreted as a form on $D_+^m = \{x \in D^m | x\mathbf{i} = x\}$.

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We now extend the form f to a skew-Hermitian form F on D^m by setting

$$F(x_1 + y_1\mathbf{j}, x_2 + y_2\mathbf{j}) = f(x_1, x_1)(\mathbf{i}-1)\mathbf{j} + f(x_1, y_2)(\mathbf{i}-1) + \\ + f(x_2, y_1)(\mathbf{i}+1) + f(y_1, y_2)(\mathbf{i}+1)\mathbf{j}$$

for all $x_1, y_1, x_2, y_2 \in D_+^m$.

The admissibility of F follows directly from the admissibility of the initial form f.

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Let $\Lambda < \mathrm{U}(F, D)$ be an arithmetic lattice. Since $D \cong M_2(K)$, we have that Λ is a type I lattice.

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On the other hand, Λ is a totally geodesic sublattice in $\Gamma < \mathrm{U}(G, D')$, where $G = \mathrm{Res}_{K/k}(F)$ is an admissible skew-Hermitian form on $(D')^{2m}$. It follows that Γ is a type II lattice.

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So we obtain that the type I orbifold $M = \mathbb{H}^{2m-1}/\Lambda$ is realised as a Weil restriction subspace in the type II orbifold $N = \mathbb{H}^{4m-1}/\Gamma$.

Trialitarian 7-dimensional orbifolds

Types of arithmetic subgroups of $\mathrm{Isom}(\mathbf{H}^n)$:

- type I (*simplest type*) are defined by quadratic forms;
- type II (only in odd dimensions n) are defined by skew-Hermitian forms;
- type III (*exceptional type* in $n = 3$ and $n = 7$).

Trialitarian 7-dimensional orbifolds

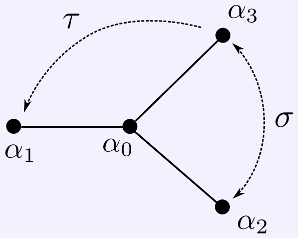

Figure: The Tits index of ${}^6D_{4,0}$ of trialitarian type.

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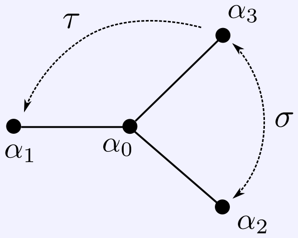

Figure: The Tits index of ${}^6D_{4,0}$ of trialitarian type.

Definition. Let $\Gamma < \mathbf{PO}_{7,1}(\mathbb{R})$ be a lattice commensurable with $\mathrm{G}(\mathscr{O})$, for some triality algebraic k-group G. Then Γ is called an *arithmetic lattice of type III*. An orbifold $M = \mathbb{H}^7/\Gamma$ is of type III if the group Γ is commensurable in the wide sense with an arithmetic lattice of type III.

Trialitarian 7-dimensional orbifolds

There are 3 possible partitions of the roots compatible with the action of the absolute Galois group of k:

$$\Delta_- = \{\alpha_0\}, \Delta_+ = \{\alpha_1, \alpha_2, \alpha_3\}; \quad (1)$$

$$\Delta_+ = \{\alpha_0\}, \Delta_- = \{\alpha_1, \alpha_2, \alpha_3\}; \quad (2)$$

$$\Delta_- = \Delta, \quad \Delta_+ = \emptyset. \quad (3)$$

This gives 3 different involutions g_1, g_2, $g_3 \in \mathrm{Comm}(\Gamma)$ which generate the Klein 4-group.

Trialitarian 7-dimensional orbifolds – applications

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Theorem 7. [Bogachev–Slavich–Sun]
All arithmetic lattices in $\mathbf{PO}_{7,1}(\mathbb{R})$ *are not LERF (locally extended residually finite).*

Trialitarian 7-dimensional orbifolds – questions

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Theorem 8. [Bergeron–Clozel]
Congruence subgroups of trialitarian lattices have vanishing first Betti numbers.

Question 1. *Are trialitarian lattices superrigid in the sense of Margulis?*

Question 2. *Do trialitarian lattices have congruence subgroup property?*