Prasad's volume formula and its applications

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IMPA

Gopal's Day

A Celebration of the 75th Birthday of Gopal Prasad
ICTS, July 30, 2020

Volume of hyperbolic manifolds

Let $\mathscr{H}^n$ be the *hyperbolic n-space*
(e.g. the upper half space with the hyperbolic metric $ds^2 = \frac{dw^2}{y^2}$).

$\mathrm{Isom}(\mathscr{H}^n)$ – the *group of isometries* of $\mathscr{H}^n$.

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We will discuss *finite volume* hyperbolic n-manifolds and orbifolds.

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For n even:

$$\mathrm{Vol}(\mathscr{M}) = \frac{\mathrm{Vol}(\mathbf{S}^n)}{2} \cdot (-1)^{n/2} \chi(\mathscr{M}) \quad \text{(Chern–Gauss–Bonnet Theorem)}$$

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If $\mathscr{M}$ is an oriented connected hyperbolic n-manifold,

$$\mathrm{Vol}(\mathscr{M}) = v_n \|\mathscr{M}\| \quad \text{(Gromov–Thurston)}$$

$\Longrightarrow$ *volume is a measure of complexity.*

Volume of hyperbolic manifolds

Maclachlan Everitt paper

CONSTRUCTING HYPERBOLIC MANIFOLDS

B. EVERITT AND C. MACLACHLAN

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ABSTRACT. The Coxeter simplex with symbol o≡o—o—o≡o is a compact hyperbolic 4-simplex and the related Coxeter group Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^4)$. The Coxeter simplex with symbol o—o—o≡o is a spherical 3-simplex, and the related Coxeter group G is the group of symmetries of the regular 120-cell. Using the geometry of the regular 120-cell, Davis [3] constructed an epimorphism $\Gamma \to G$ whose kernel K was torsion-free, thus obtaining a small volume compact hyperbolic 4-manifold $\mathbb{H}^4/K$.

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$$\bar{q}(\mathbf{x}) = x_1^2 + 2x_1x_2 + x_2^2 - x_2x_3 + x_3^2 - x_3x_4 + x_4^2 + 2x_4x_5 + x_5^2.$$

Prasad's formula

GOPAL PRASAD

Volumes of S-arithmetic quotients of semi-simple groups

Publications mathématiques de l'I.H.É.S., tome 69 (1989), p. 91-114.

Prasad's formula

VOLUMES OF S-ARITHMETIC QUOTIENTS OF SEMI-SIMPLE GROUPS

by Gopal PRASAD*

With an appendix by Moshe Jarden and Gopal Prasad

Dedicated to the memory of Harish-Chandra.

Introduction

The purpose of this paper is twofold: The first is to give a computable formula for the volumes of the S-arithmetic quotients of $G_S := \prod_{v \in S} G(k_v)$, in terms of a natural Haar measure on G_S, where G is an arbitrary absolutely quasi-simple, simply connected algebraic group defined over a global field k (i.e. a number field or the function field of a curve over a finite field) and S is a finite set of places of k containing all the archimedean ones; see § 3. The second is to use the results involved in the volume computation to provide a "good" lower (and also upper) bound for the class number of G; this is done in § 4 of the paper.

Prasad's formula

3.7. *Theorem. — We have the following*

$$\mu_S(G_S/\Lambda) = D_k^{\frac{1}{2}\dim G}(D_\ell/D_k^{[\ell:k]})^{\frac{1}{2}s(\mathscr{G})}\left(\prod_{v\in V_\infty}\left|\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right|_v\right)\tau_k(G)\ \mathscr{E};$$

where

$$\mathscr{E} = \prod_{v\in S_f}\frac{q_v^{(r_v+\dim\overline{\mathscr{M}}_v)/2}}{\#\overline{T}_v(\mathfrak{f}_v)}\cdot\prod_{v\notin S}\frac{q_v^{(\dim\overline{M}_v+\dim\overline{\mathscr{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)},$$

and $S_f = S\cap V_f$.

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where

- $\dim(G)$, r and m_i denote the dimension, rank and Lie exponents;
- l is a Galois extension of k of degree $\leqslant 3$ defined in Prasad's paper;
- $s = s(\mathscr{G})$ is an integer defined in Prasad's paper ($s = 0$ if G is an inner form of a split group and $s \geqslant 5$ if G is an outer form);
- $\tau_k(G)$ is the Tamagawa number of G over k; and
- $\mathscr{E}$ is an Euler product of the local factors $e_v = e(P_v)$.

Results about minimal volume

$H = \mathrm{PO}(n,1)^\circ = \mathrm{Isom}^+(\mathscr{H}^n)$

Theorem 1. *(B., 2004, B.–Emery, 2012) For every dimension $n \geqslant 4$ there exists a **unique** cocompact arithmetic subgroup $\Gamma_0^n < H$ of the smallest covolume. It is defined over $k_0 = \mathbb{Q}[\sqrt{5}]$ and has*

$$\mathrm{Vol}(\mathscr{H}^n/\Gamma_0^n) = \omega_c(n).$$

Theorem 2. *(B., 2004, B.–Emery, 2012) For every dimension $n \geqslant 4$ there exists a **unique** non-cocompact arithmetic subgroup $\Gamma_1^n < H$ of the smallest covolume. It is defined over $k_1 = \mathbb{Q}$ and has*

$$\mathrm{Vol}(\mathscr{H}^n/\Gamma_1^n) = \omega_{nc}(n).$$

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$$\mathrm{Vol}(\mathscr{H}^n/\Gamma_1^n) = \omega_{nc}(n).$$

$^{(*)}$ Of the first type.

$n = 2r$, r even:

$$\omega_c(n) = \frac{4 \cdot 5^{r^2+r/2} \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i);$$

$n = 2r$, r odd:

$$\omega_c(n) = \frac{2 \cdot 5^{r^2+r/2} \cdot (2\pi)^r \cdot (4r-1)}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i);$$

(B., 2004)

$n = 2r - 1$:

$$\omega_c(n) = \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)!}{2^{2r-1}\pi^r} L_{\ell_0|k_0}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i),$$

where $k_0 = \mathbb{Q}[\sqrt{5}]$ and l_0 is the quartic field with a defining polynomial $x^4 - x^3 + 2x - 1$.

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$n = 2r,\ r \equiv 0,\ 1\ (mod\ 4)$:

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$n = 2r,\ r \equiv 2,\ 3\ (mod\ 4)$:

$$\omega_{nc}(n) = \frac{2 \cdot (2^r - 1) \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \quad \text{(B.)}$$

$n = 2r-1$, r even:

$$\omega_{nc}(n) = \frac{3^{r-1/2}}{2^{r-1}} L_{\ell_1|\mathbb{Q}}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i), \text{ where } \ell_1 = \mathbb{Q}[\sqrt{-3}];$$

$n = 2r-1,\ r \equiv 1\ (mod\ 4)$:

$$\omega_{nc}(n) = \frac{1}{2^{r-2}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i);$$

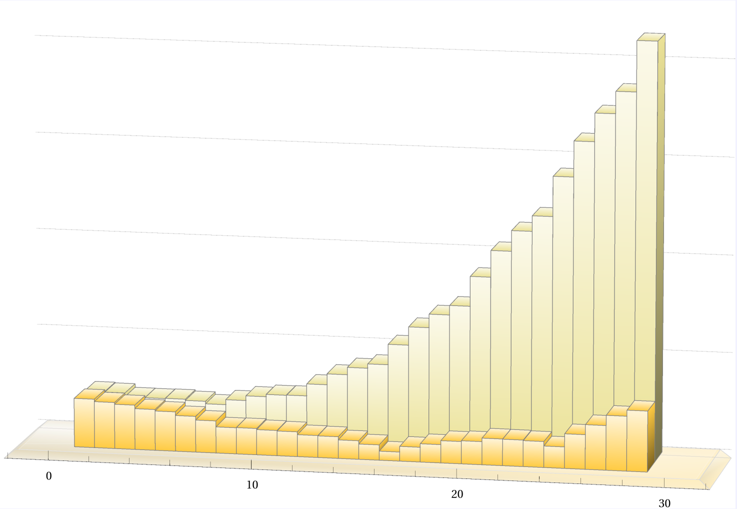
$n = 2r-1,\ r \equiv 3\ (mod\ 4)$:

$$\omega_{nc}(n) = \frac{(2^r-1)(2^{r-1}-1)}{3 \cdot 2^{r-1}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \quad \text{(B.–Emery)}$$

Proofs use

- Prasad's volume formula
- Galois cohomology of algebraic groups
- Bruhat–Tits theory
- Bounds for discriminants and class numbers (Odlyzko bounds, Brauer–Siegel theorem, Zimmert's bound for regulator)

Growth of minimal volume



*graph from ICM'2014 talk

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Emery (2014) showed that for $n > 4$ there are no compact orientable arithmetic hyperbolic n-manifolds with $\chi = 2$.

Some other results on mimimal volume

A. Salehi Golsefidy, Lattices of minimum covolume in Chevalley groups over local fields of positive characteristic. *Duke Math. J.* **146** (2009), 227–251.

V. Emery and M. Stover, Covolumes of nonuniform lattices in PU(n,1). *Amer. J. Math.* **136** (2014), 143–164.

F. Thilmany, Lattices of minimal covolume in $\mathrm{SL}_n(\mathbb{R})$. *Proc. Lond. Math. Soc.* **118** (2019), 78–102.

Subgroup growth of lattices

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Acta Math., 193 (2004), 73–104

Counting congruence subgroups

by

DORIAN GOLDFELD, ALEXANDER LUBOTZKY and LÁSZLÓ PYBER

Columbia University
New York, U.S.A.

Hebrew University
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Hungarian Academy of Sciences
Budapest, Hungary

Acta Math., 193 (2004), 105–139

Subgroup growth of lattices in semisimple Lie groups

by

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Hebrew University
Jerusalem, Israel

New College
Oxford, England, U.K.

Qualitative results

$$\mathrm{L}_H(x) = \#\{\text{conj. cls. of lattices } \Gamma < H \text{ with } \mu(H/\Gamma) < x\};$$

$$\mathrm{AL}_H(x) = \#\{ \qquad \text{arithmetic lattices} \qquad \}$$

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Theorem (H. C. Wang, 1972). *If H is not locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$, then $\mathrm{L}_H(x)$ is finite for every $x > 0$.*

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Theorem (Borel, 1981). *For $H \simeq \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$, the function $\mathrm{AL}_H(x)$ is finite for every $x > 0$.*

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Theorem (Borel, 1981). *For $H \simeq \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$, the function $\mathrm{AL}_H(x)$ is finite for every $x > 0$.*

Question. *What can we say about $\mathrm{L}_H(x)$ and $\mathrm{AL}_H(x)$ as functions of x? In particular, what is the* asymptotic behavior *of these functions?*

Motivation

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(2) connection with distributions of *primes, discriminants and class numbers* of algebraic number fields.

Theorem (Goldfeld - Lubotzky - Nikolov - Pyber'05).
Let H be a simple Lie group of real rank at least 2. Assuming the GRH and Serre's conjecture, for every lattice Γ in H the limit

$$\lim_{n\to\infty} \frac{\log s_n(\Gamma)}{(\log n)^2/\log\log n}$$

exists and equals a constant $\gamma(H)$ which depends only on H and not on Γ. The number $\gamma(H)$ is an invariant which is easily computed from the root system of H.

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Conjecture (Lubotzky et al.)**.**
Under the assumptions of the theorem

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H(x)}{(\log x)^2/\log\log x} = \gamma(H).$$

Plan

(1) Count finite index subgroups in a given lattice
(done by D. Goldfeld - A. Lubotzky - N. Nikolov - L. Pyber);

(2) Count maximal lattices;

(3) Combine (1) and (2).

Counting maximal arithmetic subgroups

Theorem 3. (B. 2007 with Appendix by Ellenberg–Venkatesh)
A. *If H contains an irreducible cocompact arithmetic subgroup (or, equivalently, H is isotypic), then there exist effectively computable positive constants A and B which depend only on the type of almost simple factors of H, such that for sufficiently large x*

$$x^A \leqslant m_H(x) \leqslant x^{B\beta(x)},$$

where $\beta(x)$ is a function which we define for an arbitrary $\varepsilon > 0$ as $\beta(x) = C(\log x)^\varepsilon$, $C = C(\varepsilon)$ is a constant which depends only on ε.

B. *If H contains a non-cocompact irreducible arithmetic subgroup then there exist effectively computable positive constants A', which depends only on the type of almost simple factors of H, and B' depending on H, such that for sufficiently large x*

$$x^{A'} \leqslant m_H^{nu}(x) \leqslant x^{B'}.$$

Growth of lattices

Theorem 4. (B.–Lubotzky, 2012)
Let H be a simple Lie group of real rank at least 2. Then

(i) *There exists a positive constant a such that $\mathrm{L}_H(x) \geqslant x^{a \log x}$ for all sufficiently large x.*

(ii) *Assuming the CSP and MP, there exists a positive constant b such that $\mathrm{L}_H(x) \leqslant x^{b \log x}$ for all sufficiently large x.*

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A crucial ingredient in the proof of part (i) of the theorem is the existence of infinite class field towers of totally real fields as established by Golod and Shafarevich.

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Open Problem. *Does $\lim\limits_{x\to\infty} \dfrac{\log \mathrm{L}_H(x)}{(\log x)^2}$ exist? And if so, what is its value?*

Note: Theorem 4 disproves Lubotzky's conjecture.

Growth of lattices

Theorem 5. (B.–Gelander–Lubotzky–Shalev, 2010)
Let $H = \mathrm{PSL}_2(\mathbb{R})$ *endowed with the Haar measure induced from the Riemanian measure of the hyperbolic plane* $\mathscr{H}^2$*. Then*

$$\lim_{x\to\infty} \frac{\log \mathrm{AL}_H(x)}{x \log x} = \frac{1}{2\pi}.$$

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Theorem 6. (BGLS, 2010)
Let $H = \mathrm{PSL}_2(\mathbb{C})$ *endowed with the Haar measure induced from the Riemanian measure of the hyperbolic space* $\mathscr{H}^3$*. Then there exist* $\alpha, \beta > 0$ *such that for* $x \gg 0$*,*

$$\alpha x \log x \leqslant \log \mathrm{AL}_H(x) \leqslant \beta x \log x.$$

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Corollary. We can extend results of Borel–Prasad (Publ. IHES, 1989), B. (Duke Math. J., 2007), and Agol–B.–Storm–Whyte (Groups, Geom., and Dynamics, 2008) to the SL_2-case.

Growth of lattices

Theorem 7. (B.–Lubotzky, 2019)
For a 2-generic simple Lie group H of real rank at least 2, we have

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H^{nu}(x)}{(\log x)^2/\log\log x} = \gamma(H),$$

where $\gamma(H)$ is an explicit constant and $\mathrm{L}_H^{nu}(x)$ is the number of conjugacy classes of non-uniform lattices in H of covolume at most x.

Here *2-generic* means that H is not of type E_6 or D_4, and if it is of type A_n, then n is of the form $n = 2^\alpha - 1$ for some $\alpha \in \mathbb{N}$.

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Conjecture 1. *Theorem 6 applies to any semisimple Lie group of real rank at least 2.*

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For a 2-generic simple Lie group H of real rank at least 2, we have

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H^{nu}(x)}{(\log x)^2/\log\log x} = \gamma(H),$$

where $\gamma(H)$ is an explicit constant and $\mathrm{L}_H^{nu}(x)$ is the number of conjugacy classes of non-uniform lattices in H of covolume at most x.

Here *2-generic* means that H is not of type E_6 or D_4, and if it is of type A_n, then n is of the form $n = 2^\alpha - 1$ for some $\alpha \in \mathbb{N}$.

Conjecture 1. *Theorem 6 applies to any semisimple Lie group of real rank at least 2.*

We prove that this conjecture is *equivalent to:*

Conjecture 2. *Fix an integer $d \geqslant 2$ and a prime l. Then for number fields k of degree d, $\mathrm{rk}_l(\mathrm{Cl}(k)) = o(\frac{\log \mathrm{D}_k}{\sqrt{\log\log \mathrm{D}_k}})$.*

(for $l = d = 2$ this follows from the Gauss theorem)

Some other results on counting lattices

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Thank You Gopal!