Counting lattices and class field towers

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Setup

H is a non-compact simple Lie group, $\mathrm{rank}_{\mathbb{R}}(\mathrm{H}) \geq 2$.

μ a Haar measure on H.

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Given a positive real number x, H contains only finitely many conjugacy classes of discrete subgroups Γ with $\mu(\mathrm{H}/\Gamma) < x$.

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3) *Combine (1) and (2)*
(M.B. – A. Lubotzky, work in progress)

Notation

$s_n(\Gamma) = \#\{\text{subgroups of } \Gamma \text{ of index at most } n\}$

$\rho^u_{\mathrm{H}}(x) = \#\{\text{conj. cls. of uniform lattices } \Gamma < \mathrm{H} \text{ with } \mu(\mathrm{H}/\Gamma) < x\}$
$\rho^{nu}_{\mathrm{H}}(x) = \#\{\text{non-uniform}\}$

$m^u_{\mathrm{H}}(x) = \#\{\text{conj. cls. of uniform max. } \Gamma < \mathrm{H} \text{ with } \mu(\mathrm{H}/\Gamma) < x\}$
$m^{nu}_{\mathrm{H}}(x) = \#\{\text{non-uniform}\}$

$\rho_{\mathrm{H}}(x) = \rho^u_{\mathrm{H}}(x) + \rho^{nu}_{\mathrm{H}}(x), \qquad m_{\mathrm{H}}(x) = m^u_{\mathrm{H}}(x) + m^{nu}_{\mathrm{H}}(x)$

Track Records

THEOREM. *(D. Goldfeld - A. Lubotzky - N. Nikolov - L. Pyber '05)* *Assuming GRH and Serres conjecture, then for every lattice* Γ *in* H *the limit*

$$\lim_{n\to\infty} \frac{\log s_n(\Gamma)}{(\log n)^2 / \log\log n}$$

exists and equals a constant $\gamma(\mathrm{H})$ *which depends only on* H *and not on* Γ. *The number* $\gamma(\mathrm{H})$ *is an invariant which is easily computed from the root system of* H.

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CONJECTURE. *(Lubotzky et al.)*
Under assumptions of the theorem

$$\lim_{x\to\infty} \frac{\log \rho_{\mathrm{H}}(x)}{(\log x)^2/\log\log x} = \gamma(\mathrm{H}).$$

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THEOREM. *(M.B., preprint)*
There exist effectively computable positive constants A, B, A', B', *which depend only on type of* H, *such that for sufficiently large* x

$$A \leq \frac{\log m^{u}_{\mathrm{H}}(x)}{\log x} \leq B\beta(x),$$

where $\beta(x) = C_\epsilon (\log x)^\epsilon$;

$$A' \leq \frac{\log m^{nu}_{\mathrm{H}}(x)}{\log x} \leq B'.$$

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CONJECTURE. $\beta(x) = const.$

New Results

THEOREM 1. *(M.B. – A. Lubotzky)*
For sufficiently large x,

$$a \leq \frac{\log \rho_{\mathrm{H}}^{u}(x)}{(\log x)^2} \leq^{*} b,$$

where $\leq^{*}$ holds if all lattices in H satisfy the congruence subgroup property; a, b are positive constants which depend only on H.

THEOREM 2. *(M.B. – A. Lubotzky)* *For sufficiently large x,*

$$a' \leq \frac{\log \rho_{\mathrm{H}}^{nu}(x)}{(\log x)^2 / \log\log x} \leq^{**} b',$$

where $\leq^{**}$ holds if H is a group of type A_{2^n-1} $(n > 1)$, B_r, C_r, D_r $(r \neq 4)$, E_7, E_8, F_4 or G_2; the positive constants a', b' depend only on H.

Some Open Problems

QUESTION.
Does $\lim_{x\to\infty} \frac{\log \rho_{\mathrm{H}}^{u}(x)}{(\log x)^2}$ exist? And if so, what is its value?

CONJECTURE. $\lim_{x\to\infty} \frac{\log \rho_{\mathrm{H}}^{nu}(x)}{(\log x)^2 / \log\log x} = \gamma(\mathrm{H})$.