On volumes of arithmetic hyperbolic n-orbifolds

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Remarks: 1. $\Gamma_0 = \Delta(2,3,7)$ is an *arithmetic subgroup* of $\mathrm{PSL}(2,\mathbb{R})$ defined over $k = \mathbb{Q}[\cos(\frac{2\pi}{7})]$.

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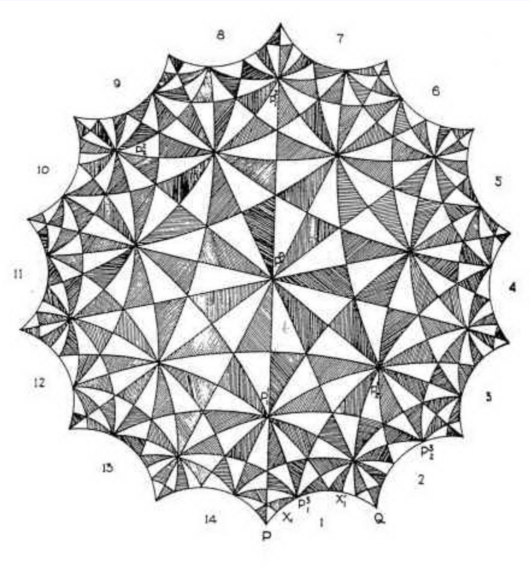
The theorem of Siegel is closely related to the Hurwitz theorem:

$\mathrm{Aut}(S_g) \leq 84(g-1)$, *$S_g$ is a Riemann surface of genus $g \geq 2$.*

The tessellation of the fundamental domain of the surface of genus 3 with 168 automorphisms by $(2,3,7)$-hyperbolic triangles:



this picture is due to Felix Klein (1879)

Non-cocompact case

can consider separately the case when Γ is non-cocompact but still has finite covolume, i.e. $\mathscr{H}/\Gamma$ has *cusps*.

Theorem 2. *There exists a discrete non-cocompact subgroup* Γ_1 *in* H *of the smallest covolume. It is unique up to conjugation and isomorphic to the modular group* $\mathrm{PSL}(2,\mathbb{Z})$*, and its covolume*

$$\mu(H/\Gamma_1) = \mathrm{vol}(\mathscr{H}^2/\Gamma_1) = \pi/3.$$

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Remarks: 1. $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$ is an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{R})$ defined over $\mathbb{Q}$.

2. $\mathrm{vol}(\mathscr{H}^2/\Gamma_1) = 4\pi\zeta(-1)$.

3. $\mathrm{vol}(\mathscr{H}^2/\Gamma_0) < \mathrm{vol}(\mathscr{H}^2/\Gamma_1)$.

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For $\mathbf{n} = \mathbf{3}$ the restricted problem was solved by Chinburg and Friedman (1986). In 2009 Ann. Math. paper Gehring and Martin announced the proof that the orbifold constructed by Chinburg and Friedman has the smallest volume among *all* hyperbolic 3-orbifolds. The non-cocompact analogue of this problem was solved by Meyerhoff (1986). Here again the minimal covolume subgroup appears to be arithmetic.

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Theorem 3. *(B.'2004, B.-Emery'2010) For every dimension $n \geq 4$ there exists a unique cocompact arithmetic subgroup $\Gamma_0^n < H$ of the smallest covolume. It is defined over $k_0 = \mathbb{Q}[\sqrt{5}]$ and has*

$$\mathrm{vol}(\mathscr{H}^n/\Gamma_0^n) = \omega_c(n).$$

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Theorem 4. *(B.'2004, B.-Emery'2010) For every dimension $n \geq 4$ there exists a unique non-cocompact arithmetic subgroup $\Gamma_1^n < H$ of the smallest covolume. It is defined over $k_1 = \mathbb{Q}$ and has*

$$\mathrm{vol}(\mathscr{H}^n/\Gamma_1^n) = \omega_{nc}(n).$$

$n = 2r$, r even:

$$\omega_c(n) = \frac{4 \cdot 5^{r^2+r/2} \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i);$$

$n = 2r$, r odd:

$$\omega_c(n) = \frac{2 \cdot 5^{r^2+r/2} \cdot (2\pi)^r \cdot (4r-1)}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i);$$

(B.'2004)

$n = 2r - 1$:

$$\omega_c(n) = \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)!}{2^{2r-1}\pi^r} L_{\ell_0|k_0}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i),$$

where $k_0 = \mathbb{Q}[\sqrt{5}]$ and l_0 is the quartic field with a defining polynomial $x^4 - x^3 + 2x - 1$.

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$n = 2r-1$, r even:

$$\omega_{nc}(n) \quad = \quad \frac{3^{r-1/2}}{2^{r-1}} L_{\ell_1|\mathbb{Q}}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i), \text{ where } \ell_1 = \mathbb{Q}[\sqrt{-3}];$$

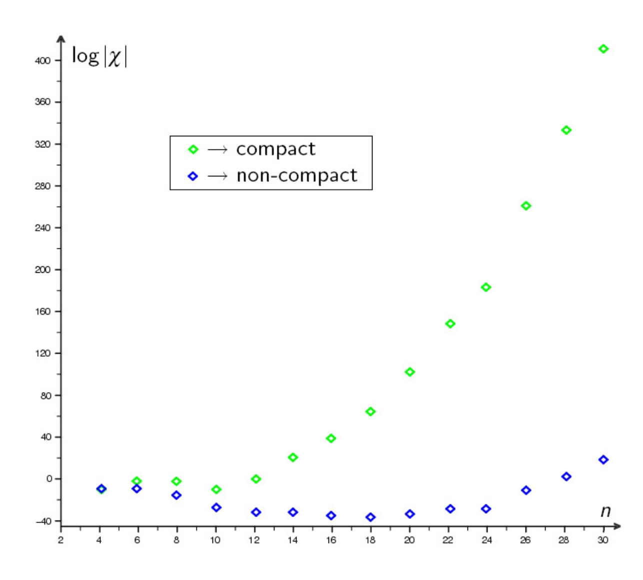
$n = 2r-1,\ r \equiv 1\ (mod\ 4)$:

$$\omega_{nc}(n) \quad = \quad \frac{1}{2^{r-2}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i);$$

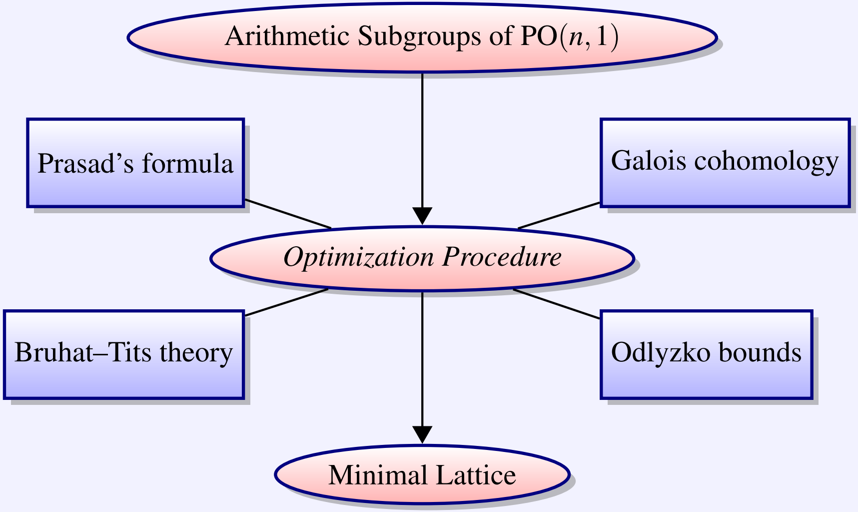
$n = 2r-1,\ r \equiv 3\ (mod\ 4)$:

$$\omega_{nc}(n) \quad = \quad \frac{(2^r-1)(2^{r-1}-1)}{3 \cdot 2^{r-1}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \quad \text{(B.-Emery)}$$

Growth of minimal covolume



Ingredients of the proof:



Skip details

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for $v_0 \in V_\infty(k)$: $\mathrm{G}(k_{v_0}) \cong \mathrm{SO}(n,1)$;
for $v \in V_\infty \setminus \{v_0\}$: $\mathrm{G}(k_v) \cong \mathrm{SO}(n+1)$.

$\Lambda = \mathrm{G}(k) \cap \prod_{v \in V_f} \mathrm{G}(k_v)$ is a *principal arithmetic subgroup* of G.

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Assume n even.

Let μ^{EP} is the *Euler-Poincaré measure* in the sense of Serre. Then

$$|\chi(\mathrm{G}/\Lambda)| = \mu^{EP}(\mathrm{G}/\Lambda)$$

We can compute $\mu^{EP}(\mathrm{G}/\Lambda)$ using *Prasad's volume formula*.

Prasad's formula for $\mathrm{SO}(n,1)$*, n even*

$$\mu^{EP}(\Lambda\backslash G) = 2\mathscr{D}_k^{\frac{1}{2}\dim G}\left(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]}\tau_k(G)\,\mathscr{E}\prod_{v\in T}\lambda_v,$$

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- $\mathscr{D}_k$ is the discriminant of k;
- $r = n/2$, the absolute rank of G;
- dimension $\dim G = 2r^2 + r$ and Lie exponents $m_i = 2i-1$;
- the Tamagawa number $\tau_k(G) = 2$;
- $\mathscr{E}$ is an Euler product which in our case is given by $\mathscr{E} = \zeta_k(2)\cdot\ldots\cdot\zeta_k(2r)$;
- $\lambda_v \in \mathbb{Q}$ are local densities, v runs through a finite subset $T \subset V_f$.

Assume $r = n/2$ *is large* (e.g. $r \geq 30$) and n *is even*

$$|\chi(\mathrm{G}/\Lambda)| = C\mathscr{D}_k^{r^2+r/2}\left(\prod_{i=1}^{r}\frac{(2i-1)!}{(2\pi)^{2i}}\right)^{[k:\mathbb{Q}]}\mathscr{E}\prod_{v\in T}\lambda_v$$

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This bound grows super-exponentially with n and attains its minimum on $k = \mathbb{Q}$.

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Odd dimensional case is more complicated. Moreover, low dimensions, maximal arithmetic subgroups and precise formulas require much more care!

Differences btw. odd and even dimensions:

$n = 2r$ – even	$n = 2r - 1$ – odd
type B_r	type D_r
only inner forms	inner and outer forms
$\lvert Z(\mathrm{Spin}(n,1))\rvert = 2$	$\lvert Z(\mathrm{Spin}(n,1))\rvert = 4$

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The main features of the results obtained:

- *Precise computation of the minimal covolume;*
- *Uniqueness of the extremal subgroups.*

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