Euler-Poincaré characteristic of arithmetic hyperbolic n-orbifolds

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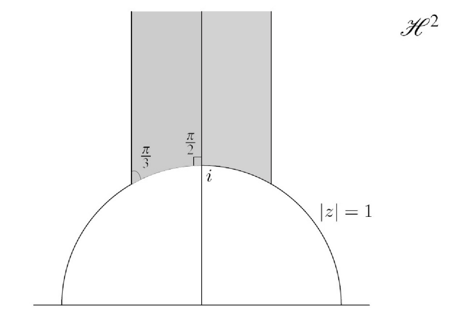

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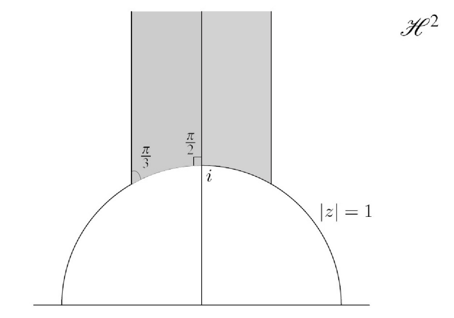

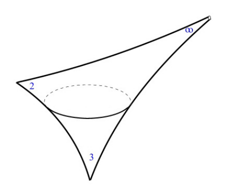

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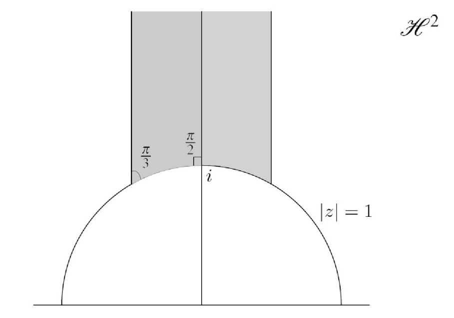

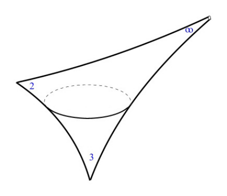

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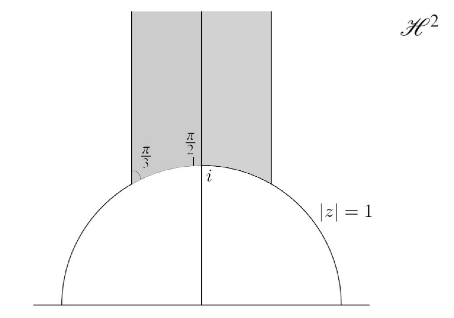

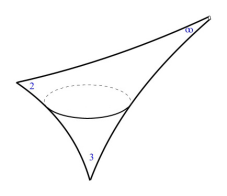

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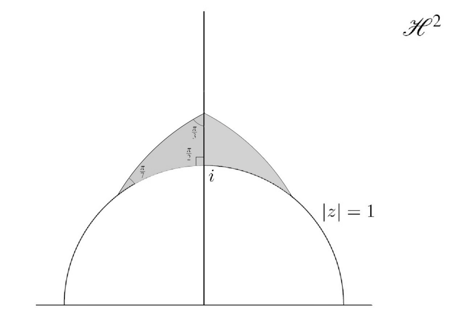

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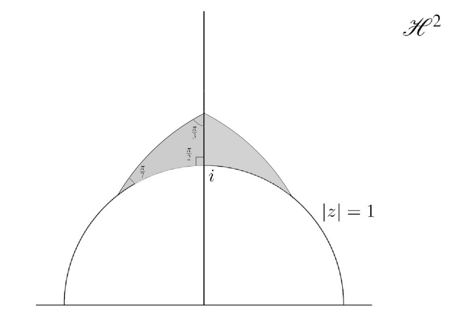

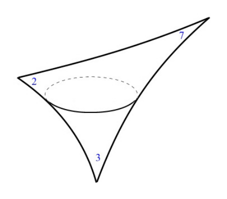

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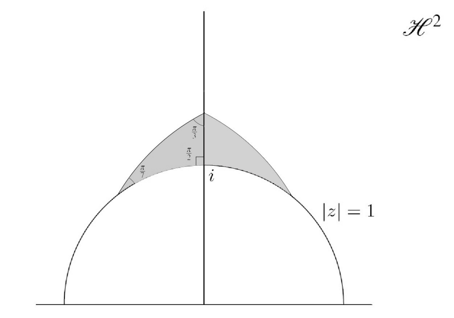

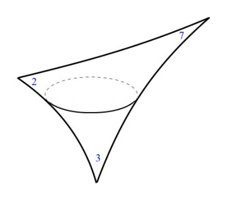

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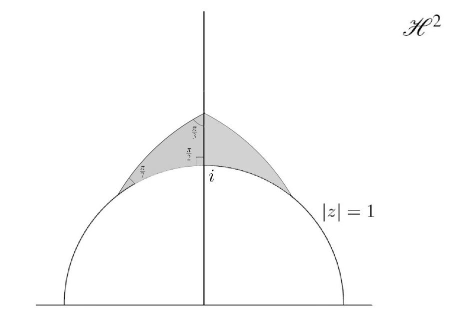

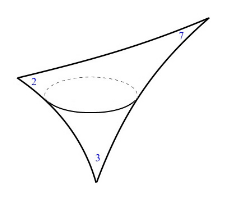

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$\chi(\mathscr{O}_2) = -1/42 = 2^{-1}\zeta_k(-1)$

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QUESTION. WHAT HAPPENS IN HIGHER DIMENSIONS ?

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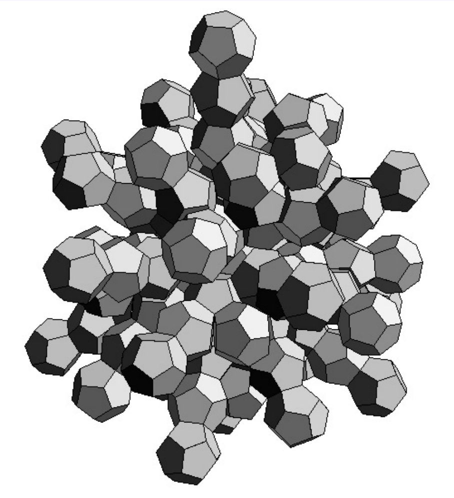
120-cell in $\mathscr{H}^4$:



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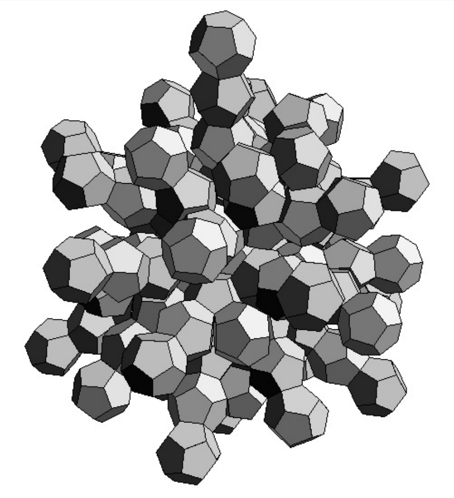

M. Davis: One can identify opposite dodecahedral faces of the 120-cell to obtain a compact hyperbolic 4-manifold $\mathscr{M}$ *.*

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This reduces the problem to a finite computation

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NOTE: This is **certainly not** the case for high dimensions if restricted to arithmetic manifolds and most likely also for arbitrary manifolds.

n-dimensional case

THEOREM. *(M.B.) Let $n = 2r \geq 4$.*
A. *There exists a* **unique compact minimal** *arithmetic n-orbifold $\mathcal{O}^n_{min}$. It is defined over the field $k = \mathbb{Q}[\sqrt{5}]$ and has Euler characteristic*

$$|\chi(\mathcal{O}^n_{min})| = \frac{\lambda(r)}{4^{r-1}} \prod_{i=1}^{r} |\zeta_k(1-2i)|,$$

$\lambda(r) = 1$ if r is even and $\lambda(r) = \frac{4^r-1}{2}$ if r is odd.

B. *There exists a* **unique non-compact minimal** *arithmetic n-orbifold $\mathcal{O}'^n_{min}$ which is defined over $\mathbb{Q}$ and has*

$$|\chi(\mathcal{O}'^n_{min})| = \frac{\lambda'(r)}{2^{r-2}} \prod_{i=1}^{r} |\zeta(1-2i)|,$$

$\lambda'(r) = 1$ if $r \equiv 0, 1 \pmod 4$ and $\lambda'(r) = \frac{2^r-1}{2}$ if $r \equiv 2, 3 \pmod 4$.

remarks

- the field of definition of the smallest compact 2-orbifold is the cubic field $\mathbb{Q}[\cos(\frac{2\pi}{7})]$, starting from $n = 4$ the field switch to $\mathbb{Q}[\sqrt{5}]$ and stabilize

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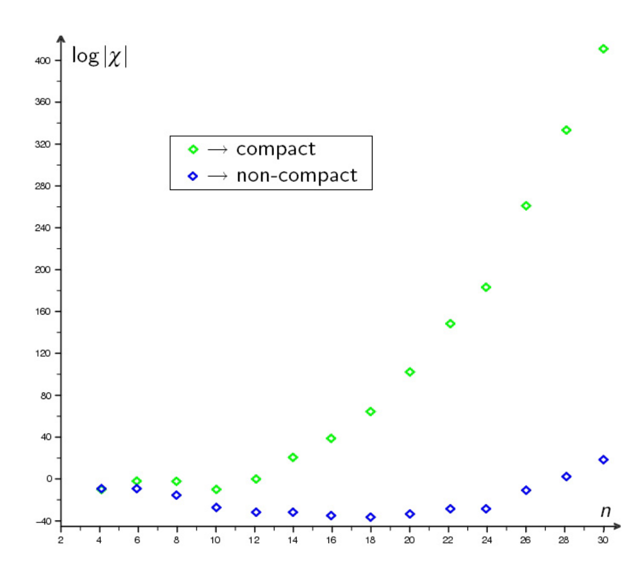

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Let k be a number field, G – an algebraic group $/k$ such that
for $v_0 \in V_\infty(k)$: $\mathrm{G}(k_{v_0}) \cong \mathrm{SO}(1,n)$;
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We can compute $\mu^{EP}(\mathrm{G}/\Lambda)$ using Prasad's volume formula.

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Prasad's formula:

$$\mu^{EP}(\Lambda\backslash G) = 2\mathscr{D}_k^{\frac{1}{2}\dim G}\left(\prod_{i=1}^{r}\frac{m_i!}{(2\pi)^{m_i+1}}\right)^{[k:\mathbb{Q}]}\tau_k(G)\,\mathscr{E}\prod_{v\in T}\lambda_v,$$

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- $\mathscr{D}_k$ is the discriminant of k;
- $r = n/2$, the absolute rank of G;
- dimension $\dim G = 2r^2 + r$ and Lie exponents $m_i = 2i - 1$;
- the Tamagawa number $\tau_k(G) = 2$;
- $\mathscr{E}$ is an Euler product which in our case is given by $\mathscr{E} = \zeta_k(2) \cdot \ldots \cdot \zeta_k(2r)$;
- $\lambda_v \in \mathbb{Q}$ are local densities in v from finite set $T \subset V_f$

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low dimensions, maximal arithmetic subgroups and precise formulas require much more care

references

[1] M. Belolipetsky, On volumes of arithmetic quotients of $\mathrm{SO}(1, n)$, *arXiv: math.NT/0306423*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **3** (2004), 749–770.

[2] M. Belolipetsky, Addendum to: On volumes of arithmetic quotients of $\mathrm{SO}(1, n)$, *arXiv: math.NT /0610177*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., to appear.*