Hyperbolic orbifolds of small volume

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IMPA

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Volume in hyperbolic geometry

$\mathscr{H}^n$ – the *hyperbolic n-space*
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We will discuss *finite volume* hyperbolic n-manifolds and orbifolds.

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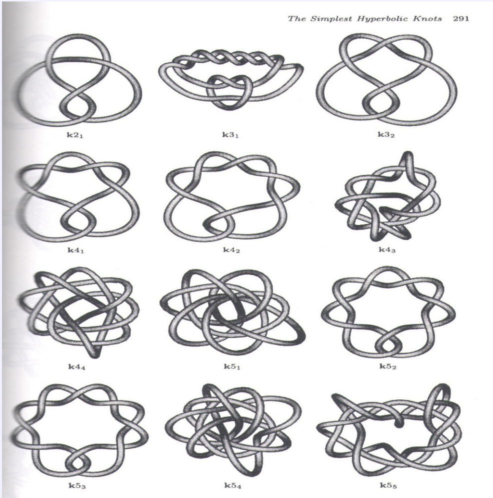
If $\mathscr{M}$ is an oriented connected hyperbolic n-manifold,

$$\mathrm{Vol}(\mathscr{M}) = v_n \|\mathscr{M}\| \quad \text{(Gromov–Thurston)}$$

$\Longrightarrow$ *volume is a measure of complexity.*

Volume in hyperbolic geometry



(Callahan–Dean–Weeks'1999)

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The problem (restricted to arithmetic manifolds) is connected with difficult open problems in number theory about rational independence of certain Dedekind ζ-values.

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- (Kazhdan–Margulis, 1968) Proved the existence of the lower bound in general.
- (B., B.–Emery) Minimal volume *arithmetic* hyperbolic n-orbifolds for $n \geqslant 4$.

Arithmeticity and volume: Example

$\mathscr{H}^2$ – the hyperbolic plane with the Poincaré metric.

$\mathrm{Isom}^+(\mathscr{H}^2) = \mathrm{PSL}(2,\mathbb{R})$.

$\Gamma = \mathrm{PSL}(2,\mathbb{Z}) < \mathrm{PSL}(2,\mathbb{R})$, a discrete subgroup.

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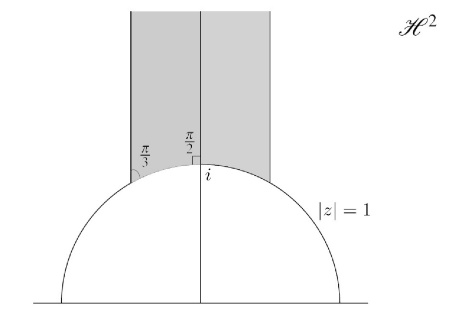

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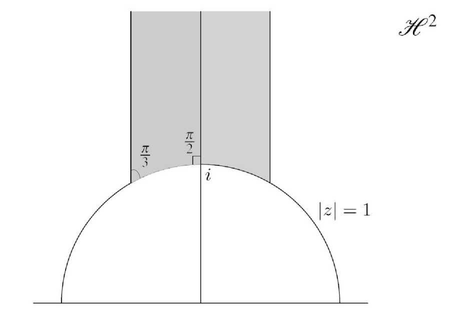

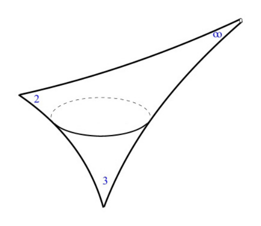

$$\mathrm{Vol}(\mathscr{O}) = \iint_{\mathscr{F}} \frac{dx\,dy}{y^2} = -2\pi\chi(\mathscr{O})$$

$$= \frac{1}{\pi}\prod_{\text{primes}} \frac{p^3}{\#\mathrm{PSL}_2(\mathbb{F}_p)} = 4\pi|\zeta(-1)| = \frac{\pi}{3}.$$

Arithmeticity and volume: Definitions

Let G be an algebraic group defined over a number field k.

Let $\mathrm{P} = (\mathrm{P}_v)_{v \in V_\mathrm{f}}$ a collection of parahoric subgroups $\mathrm{P}_v \subset \mathrm{G}(k_v)$, where v runs through all finite places of k and k_v denotes the non-archimedean completion of the field. The family P is called *coherent* if $\prod_{v \in V_\mathrm{f}} \mathrm{P}_v$ is an open subgroup of the finite adèle group $\mathrm{G}(\mathbb{A}_f(k))$. The group

$$\Lambda = \mathrm{G}(k) \cap \prod_{v \in V_\mathrm{f}} \mathrm{P}_v$$

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Every *maximal* arithmetic subgroup is a normalizer of a principal arithmetic subgroup.

Groups versus covers

If $\Gamma_1 < \Gamma_0$, then

$$\begin{array}{c} \mathscr{O}_1 = \mathscr{H}^n/\Gamma_1 \\ \Big\downarrow \text{\textit{cover}} \\ \mathscr{O}_0 = \mathscr{H}^n/\Gamma_0 \end{array}$$

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Corollary. Minimal volume orbifolds correspond to maximal discrete subgroups.

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Borel–Harish-Chandra Theorem. Arithmetic subgroups are discrete and have finite covolume.

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- G. Harder, A Gauss–Bonnet formula for discrete arithmetically defined groups (Ann. Sci. École Norm. Sup., 1971)
- A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds (Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1981)
- *G. Prasad, Volumes of S-arithmetic quotients of semi-simple groups (Inst. Hautes Études Sci. Publ. Math., 1989)*
- B. Gross, On the motive of a reductive group (Invent. Math., 1997)

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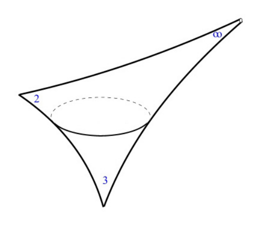

$$\mathscr{O} = \mathscr{H}^2/\mathrm{PSL}(2,\mathbb{Z})$$

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Results about minimal volume

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Theorem 1. *(B.'2004, B.–Emery'2012) For every dimension $n \geqslant 4$ there exists a **unique** cocompact arithmetic subgroup $\Gamma_0^n < H$ of the smallest covolume. It is defined over $k_0 = \mathbb{Q}[\sqrt{5}]$ and has*

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Theorem 2. *(B.'2004, B.–Emery'2012) For every dimension $n \geqslant 4$ there exists a **unique** non-cocompact arithmetic subgroup $\Gamma_1^n < H$ of the smallest covolume. It is defined over $k_1 = \mathbb{Q}$ and has*

$$\mathrm{Vol}(\mathscr{H}^n/\Gamma_1^n) = \omega_{nc}(n).$$

$n = 2r$, r even:

$$\omega_c(n) = \frac{4 \cdot 5^{r^2+r/2} \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i);$$

$n = 2r$, r odd:

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$n = 2r - 1$:

$$\omega_c(n) = \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r-1)!}{2^{2r-1}\pi^r} L_{\ell_0|k_0}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i),$$

where $k_0 = \mathbb{Q}[\sqrt{5}]$ and l_0 is the quartic field with a defining polynomial $x^4 - x^3 + 2x - 1$.

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$$\omega_{nc}(n) = \frac{1}{2^{r-2}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i);$$

$n = 2r-1,\ r \equiv 3\ (mod\ 4)$:

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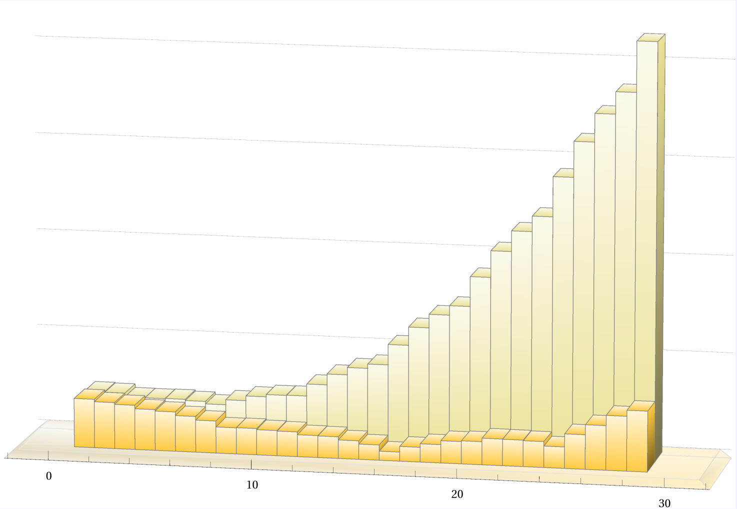
Proofs use

- Prasad’s volume formula
- Galois cohomology of algebraic groups
- Bruhat–Tits theory
- Bounds for discriminants and class numbers (Odlyzko bounds, Brauer–Siegel theorem, Zimmert’s bound for regulator)

Growth of minimal volume



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- *For* $n \geqslant 5$ *we have* $\omega_c(n) > \omega_{nc}(n)$ *(*"compact > open"*).*

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Conjecture. *(B.–Emery) Let $\mathscr{M}$ be a compact hyperbolic manifold of dimension $n \neq 3$. Then there exists a noncompact hyperbolic n-manifold $\mathscr{N}$ whose volume is smaller than the volume of $\mathscr{M}$.*

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The conjecture is *true* for

$n = 2$ – easy

$n = 4$ – follows from Ratcliffe–Tschantz'2000

$n = 6$ – follows from Everitt–Ratcliffe-Tschantz'2012

arithmetic manifolds of dimension $n \geqslant 30$ (B.–Emery'2013)

Minimal volume without arithmeticity

Lemma. *(Margulis) For every dimension n there is a constant $\mu = \mu_n > 0$ such that for every discrete group $\Gamma < \mathrm{Isom}(\mathscr{H}^n)$ and every $x \in \mathscr{H}^n$, the group*

$$\Gamma_\mu(x) = \langle \gamma \in \Gamma \mid \mathrm{dist}(x, \gamma(x)) \leqslant \mu \rangle$$

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Theorem. *(Gelander) Given a hyperbolic n-orbifold $\mathscr{O}^n$, we have*

$$\mathrm{Vol}(\mathscr{O}^n) \geqslant \frac{2v(0.25\varepsilon)^2}{v(1.25\varepsilon)}, \quad \varepsilon = \min\{\frac{\mu_n}{10}, 1\}.$$

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Proposition. *There exists a constant $C > 0$ such that $\mu_n \leqslant \frac{C}{\sqrt{n}}$.*

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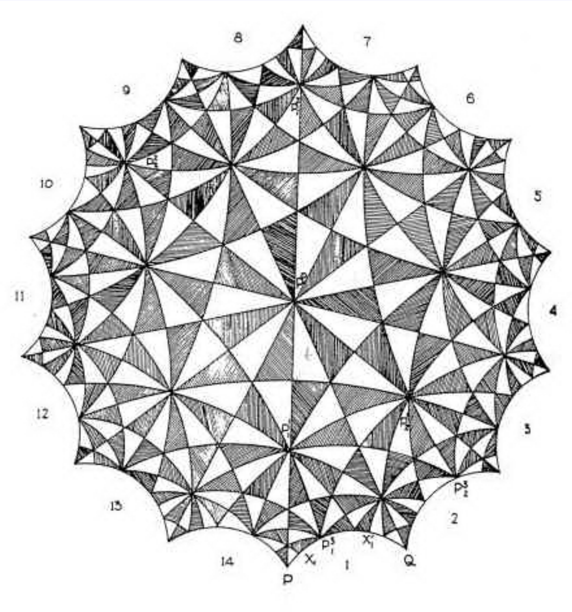
It follows from the conjecture that we expect the minimal volume to *grow super-exponentially* but so far we can prove only *super-exponentially decreasing* bounds!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