Growth of lattices

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Introduction

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Examples
Let $H = \mathrm{PO}(n,1)^o$ and $K = \mathrm{SO}(n)$.
Then $X = K\backslash H \simeq \mathbb{H}^n$ is the hyperbolic n-space. If $\Gamma < H$ is a lattice, then X/Γ is a finite volume hyperbolic n-orbifold.

a lattice in $\mathrm{Isom}(\mathbb{H}^2)$:

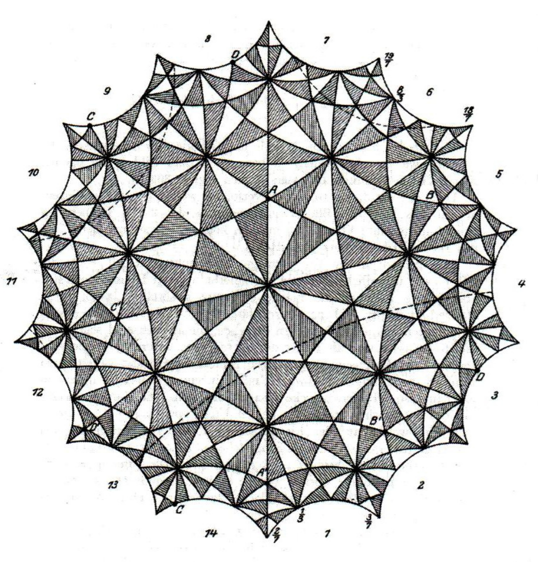

image from F. Klein's paper (1879)

a lattice in Isom($\mathbb{H}^3$):

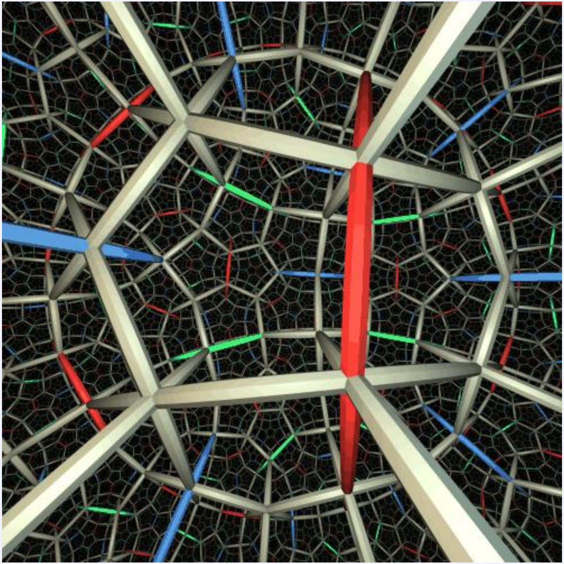

image from W. Thurston's book

Arithmeticity and commensurability

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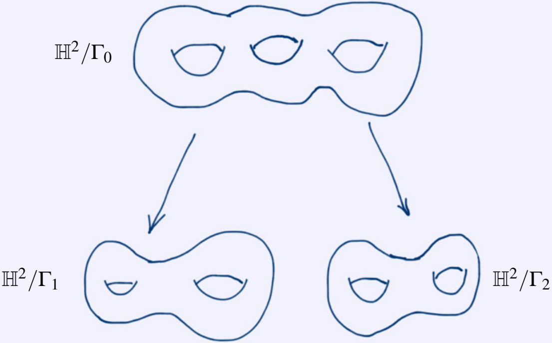

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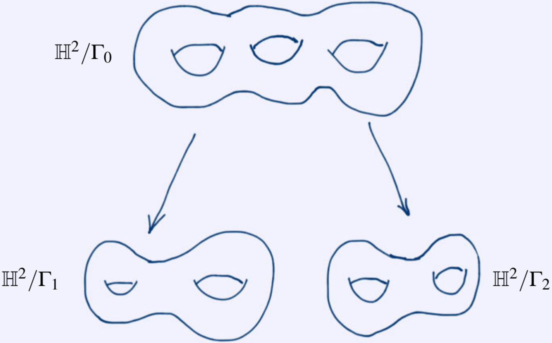

Definition. Subgroups Γ_1 and Γ_2 of H are called *commensurable* if $\Gamma_0 = \Gamma_1 \cap \Gamma_2$ has finite index in Γ_1 and Γ_2, i.e. if X/Γ_0 and X/Γ_1 have a common finite sheet cover.

Commensurator $\mathrm{Comm}_H(\Gamma) = \{h \in H \mid [\Gamma : \Gamma \cap h^{-1}\Gamma h] < \infty\}$.

arithmeticity and commensurability

Theorem (Margulis). *A discrete subgroup* $\Gamma \leqslant H$ *is arithmetic if and only if* $\mathrm{Comm}_H(\Gamma)$ *is dense in* H.

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Example. Let $H = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Then $\mathrm{Comm}_H(\Gamma) = \mathrm{SL}_n(\mathbb{Q})$, and it is dense in H.

Constructing non-arithmetic lattices is much harder!

qualitative results

$$\mathrm{L}_H(x) = \#\{\text{conj. cls. of lattices } \Gamma < H \text{ with } \mu(H/\Gamma) < x\};$$
$$\mathrm{AL}_H(x) = \#\{ \qquad \text{arithmetic lattices} \qquad \}$$

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Theorem (Borel'81). *For $H \simeq \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PSL}_2(\mathbb{C})$, the function $\mathrm{AL}_H(x)$ is finite for every $x > 0$.*

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Question. *What can we say about $\mathrm{L}_H(x)$ and $\mathrm{AL}_H(x)$ as functions of x? In particular, what is the* asymptotic behavior *of these functions?*

motivation

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(2) connection with distributions of *primes, discriminants and class numbers* of algebraic number fields.

previous work

Theorem (Burger - Gelander - Lubotzky - Mozes'02).
Let $H = \mathrm{PO}(m,1)$, $m \geqslant 4$. *There exist positive real numbers* $a = a(m)$ *and* $b = b(m)$ *such that for all* $x \gg 0$,

$$ax \log x \leqslant \log \mathrm{L}_H^{\circ}(x) \leqslant bx \log x,$$

where $\mathrm{L}_H^{\circ}(x)$ *is the number of conjugacy classes of torsion free lattices of covolume at most* x.

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Theorem (Goldfeld - Lubotzky - Nikolov - Pyber'05).
Let H be a simple Lie group of real rank at least 2. Assuming the GRH and Serre's conjecture, for every lattice Γ in H the limit

$$\lim_{n\to\infty} \frac{\log s_n(\Gamma)}{(\log n)^2/\log\log n}$$

exists and equals a constant $\gamma(H)$ which depends only on H and not on Γ. The number $\gamma(H)$ is an invariant which is easily computed from the root system of H.

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Conjecture. (Lubotzky et al.)
Under the assumptions of the theorem

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H(x)}{(\log x)^2/\log\log x} = \gamma(H).$$

results

Theorem 1. (B.-Gelander-Lubotzky-Shalev, 2010)
Let $H = \mathrm{PSL}_2(\mathbb{R})$ *endowed with the Haar measure induced from the Riemanian measure of the hyperbolic plane* $\mathbb{H}^2$*. Then*

$$\lim_{x\to\infty} \frac{\log \mathrm{AL}_H(x)}{x \log x} = \frac{1}{2\pi}.$$

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Theorem 2. (BGLS, 2010)
Let $H = \mathrm{PSL}_2(\mathbb{C})$ *endowed with the Haar measure induced from the Riemanian measure of the hyperbolic space* $\mathbb{H}^3$*. Then there exist* $\alpha, \beta > 0$ *such that for* $x \gg 0$*,*

$$\alpha x \log x \leqslant \log \mathrm{AL}_H(x) \leqslant \beta x \log x.$$

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Corollary. We can extend results of Borel-Prasad (Publ. IHES, 1989), B. (Duke Math. J., 2007), and Agol-B.-Storm-Whyte (Groups, Geom., and Dynamics, 2008) to the SL_2-case.

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Crucial Facts:

(1) Growth of lattices is dominated by the subgroup growth of Γ*;*

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Hence

$$s_{\frac{x}{-2\pi\chi(\Gamma)}}(\Gamma) = \left(\frac{x}{-2\pi\chi(\Gamma)}\right)^{(-\chi(\Gamma)\frac{1}{-2\pi\chi(\Gamma)}+o(1))x}$$

on the proof of Theorem 1 (continue)

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This 'almost' proves Theorem 1, except that the *error term* $o(1)$ depends on Γ! This can potentially change the asymptotic which indeed happens in the higher rank Lie groups. One of the main results of BGLS is a proof of the strong upper bounds for the error term. This is achieved with a combination of *algebraic, arithmetic, and geometric techniques*.

results

Theorem 3. (B.-Lubotzky, 2012)
Let H be a simple Lie group of real rank at least 2. Then

(i) *There exists a positive constant a such that $\mathrm{L}_H(x) \geqslant x^{a \log x}$ for all sufficiently large x.*

(ii) *Assuming the CSP and MP, there exists a positive constant b such that $\mathrm{L}_H(x) \leqslant x^{b \log x}$ for all sufficiently large x.*

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Problem. *Does* $\lim_{x\to\infty} \frac{\log \mathrm{L}_H(x)}{(\log x)^2}$ *exist? And if so, what is its value?*

Note: Theorem 3 disproves Lubotzky's conjecture.

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$\mathrm{IL}_H(x) \quad = \#\{\text{conj. cls. isospectral lattices } \Gamma < H \text{ with } \mu(H/\Gamma) < x\}$

Theorem 4. (B.-Linowitz, 2016)
Let H be a simple Lie group of real rank at least 2. Then there exists a positive constant $c = c(H)$ such that

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and a bound of the same shape holds for torsion-free lattices.

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Conjecture. *There exists $c = c(H) > 0$ such that $\mathrm{IL}_H(x) \geqslant x^{c \log x}$, i.e. the same growth type as the total number of lattices!*

results

Theorem 5. (B.-Lubotzky, 2017)
For a 2-generic simple Lie group H of real rank at least 2, we have

$$\lim_{x\to\infty} \frac{\log \mathrm{L}_H^{nu}(x)}{(\log x)^2/\log\log x} = \gamma(H),$$

where $\gamma(H)$ is an explicit constant and $\mathrm{L}_H^{nu}(x)$ is the number of conjugacy classes of non-uniform lattices in H of covolume at most x.

Here *2-generic* means that H is not of type E_6 or D_4, and if it is of type A_n, then n is of the form $n = 2^\alpha - 1$ for some $\alpha \in \mathbb{N}$.

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We prove that this conjecture is *equivalent to:*

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(for $l = d = 2$ this follows from the Gauss theorem)

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Thank You