Some computational problems from geometry of lattices

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Plan

1. *Automorphism groups*
2. *Minimal volume*
3. *Growth of lattices*
4. *Arithmetic reflection groups*

Notations

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Examples:

- $H = \mathrm{PSL}(2,\mathbb{R})$, $\mathscr{X} = H/\mathrm{PSO}(2)$ is the hyperbolic plane $\mathscr{H}^2$, the loc. sym. spaces are Riemann surface (possibly with singularities)
- $H = \mathrm{PO}(n,1)$, $\mathscr{X} = H/\mathrm{PO}(n) = \mathscr{H}^n$, the loc. sym. spaces are hyperbolic n-manifolds and orbifolds

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Theorem 1.1. *(MB – A. Lubotzky'2005) For every $n \geq 2$ and every finite group G there exist infinitely many compact n-dimensional hyperbolic manifolds $\mathscr{M}$ with $\mathrm{Aut}(\mathscr{M}) \cong G$.*

(This was known before for $n = 2$ (Greenberg'1974) and $n = 3$ (Kojima'1988).)

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Problem 1.2. *Given G describe explicitly at least one such $\mathscr{M}$.*

Discussion:

- The proof of Thm. 1.1 is non constructive. It uses:
  (a) Gromov–Piatetskii-Shapiro interbreeding construction of non-arithmetic lattices;
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- Can do arithmetic instead (MB – C. Leininger, unpublished).
- Then can possibly use effective strong approximation. If this works, subgroup growth should provide a computable upper bound for $\mu(\mathscr{M})$.

Extremal casees:

- $n = 2$, $G = \{e\}$ — known, see [B. Everitt, Glasgow Math. J. 39 (1997), 221–225].
- $n = 2$, G a Hurwitz group (maximal symmetry) — see e.g. [M. Conder, An update on Hurwitz groups, preprint].
  The least genus of such $\mathscr{M}$ is 3, and the corresponding Riemann surface is the Klein quartic.
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Problem 1.2(a). *Find possible higher dimensional analogues of these results.*

Two general questions:

Question 1.3. *Does Thm. 1.1 generalise to the complex hyperbolic case ($H = \mathrm{PU}(n,1)$)?*

Question 1.4. *Given arbitrary $\mathscr{X}$ does there exist an associated asymmetric manifold $\mathscr{M}$?*

(Not true for an arbitrary automorphism group G, the congruence subgroup property can be an obstruction.)

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Example: If $\mathscr{X} = \mathscr{H}^2$, $\mathscr{M}_{min}$ is the unique compact arithmetic orbifold corresponding to $\Gamma = (2,3,7)$, the Hurwitz triangle group. The non-compact $\mathscr{M}_{min} = \mathrm{PSL}(2,\mathbb{Z})\backslash\mathscr{H}^2$ which is also arithmetic. The smallest non-arithmetic orbifold is given by $\Gamma = (2,3,13)$ (see MB'1997).

For $\mathscr{X} = \mathscr{H}^n$, $\mathscr{M}_{min}$ — arithmetic hyperbolic n-orbifold the problem is solved:

Chinburg – Friedman'1986 ($n = 3$); MB'2004 ($n \geq 4$, even); MB – V. Emery, preprint ($n \geq 5$, odd).

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The problem of finding the smallest arithmetic hyperbolic n-manifold remains open for all $n \geq 4$ in the compact case and for $n = 5$ or $n \geq 7$ in the non-compact case.

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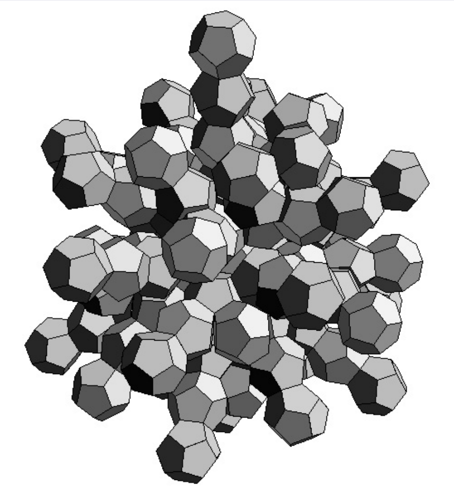
- C. Long [Bull. Lond. Math. Soc. 40 (2008), 913–916] produced other eight examples with $\chi = 16$.

Davis' example:

Can identify opposite dodecahedral faces of the 120-cell in $\mathscr{H}^4$ to obtain a compact hyperbolic manifold $\mathscr{M}$.



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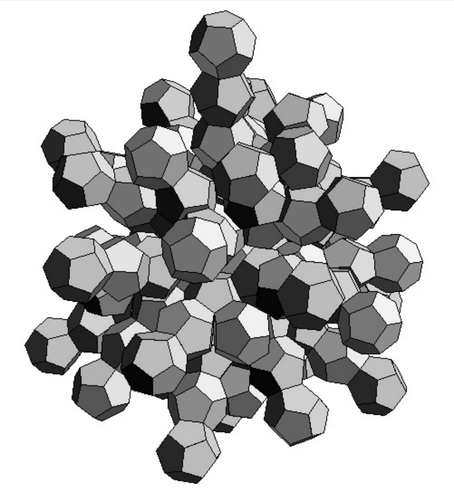

Simpler Problem: Is it the 4-dimensional analogue of the Klein quartic?

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Theorem 3.1. *(MB – Gelander – Lubotzky – Shalev'2008)* *Let $H = \mathrm{PSL}(2,\mathbb{R})$ endowed with the Haar measure induced from the Riemanian measure of the hyperbolic plane $\mathscr{H}^2$. Then*

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Theorem 3.2. *(MB – Lubotzky'2009)*
Let H be a simple Lie group of real rank at least 2. Then there exist constants a, $b > 0$ such that

$$x^{a\log x} \leq \mathrm{AL}_H(x) \leq x^{b \log x}$$

for all $x \geq X_0$.

Problem 3.3. *Compute a, b and X_0 for given H, μ.*

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Problem 3.5. *What can we say about the behavior of the function $\mathrm{AL}_H(x)$ for small values of x?*

Some results on Probl. 3.5:

- MB [Duke Math. J. 140 (2007), 1–33] give information about the smallest value of x for which $\mathrm{AL}_H(x)$ is non-zero (i.e., the minimal volume) in a general setting.

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- Maclachlan – Rosenberger [Comensurability classes of arithmetic Fuchsian surface groups of genus 2, preprint] give complete description of the comm. classes for signature (2;–). This again can be used to get bounds on $\mathrm{AL}_H(x)$ for $H = \mathrm{PSL}(2,\mathbb{R})$ and small x.

4. Arithmetic reflection groups

Some history:

Theorem 4.1. *(Vinberg'1981) Arithmetic hyperbolic reflection groups do not exist in dimensions* ≥ 30*.*

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What remained was to understand the general picture for the small dimensions.

Recent results:

Theorem 4.3. *(Long–Maclachlan–Reid'2005) The number of maximal arithmetic reflection groups is finite in dimension* 2*.*

Theorem 4.4. *(Agol'2005) The number of maximal arithmetic reflection groups is finite in dimension* 3*.*

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Theorem 4.5. *(Agol–MB–Storm–Whyte; Nikulin'∼2006) There are only finitely many maximal arithmetic hyperbolic reflection groups in all dimensions.*

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Question 4.7. *Does there exist any maximal arithmetic hyperbolic reflection group which is not congruence?*