Arithmetic Hyperbolic Reflection Groups

Mikhail Belolipetsky, IMPA

Groups

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$$\textbf{GROUPS} \quad \longleftrightarrow \quad \textbf{SYMMETRY}$$

a snowflake:

photo by K. Libbrecht

Crystallographic groups

Let $X = \mathbb{E}^n$ be the Euclidean space of dimension n.

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A **fundamental domain** for the action of Γ on X is a closed subset $R \subset X$ such that

(i) $\bigcup_{g \in \Gamma} g(R) = X$, and

(ii) $\mathring{R} \cap g(\mathring{R}) = \emptyset$ for all non-trivial $g \in \Gamma$, where $\mathring{R}$ is the interior of R.

Any crystallographic group has a fundamental domain, which is a **polyhedron** in $\mathbb{E}^n$.

17 crystallographic groups in dimension 2:

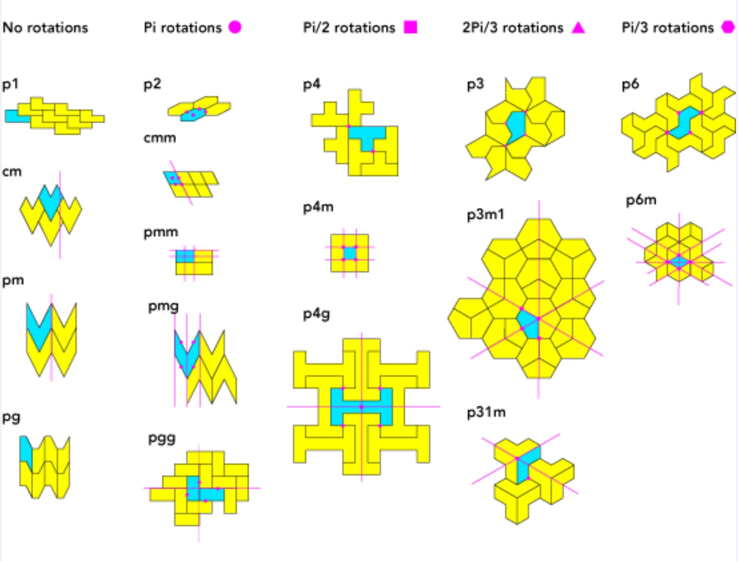

image from `http://web.media.mit.edu`

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n	# of groups	
4	4783	(Brown, Bülow, Neubüser et al., 1978)
5	222018	(Plesken and Schulz, 2000)
6	28934974	(Plesken and Schulz, 2000)
$\geqslant 7$	?, the number grows with n	

A cubic lattice in dimension 3:

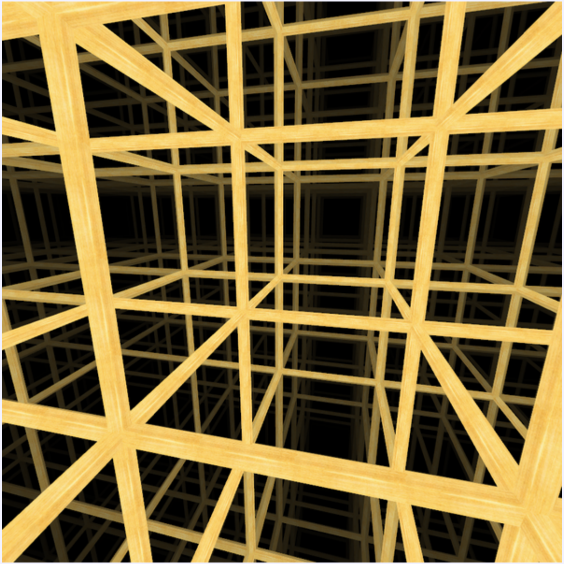


image from wikipedia

A cubic lattice in dimension 4:

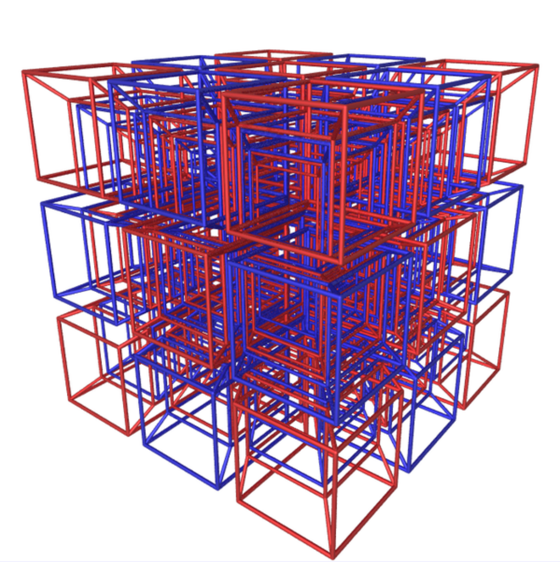


image from wikipedia

A right-angled hyperbolic tessellation:

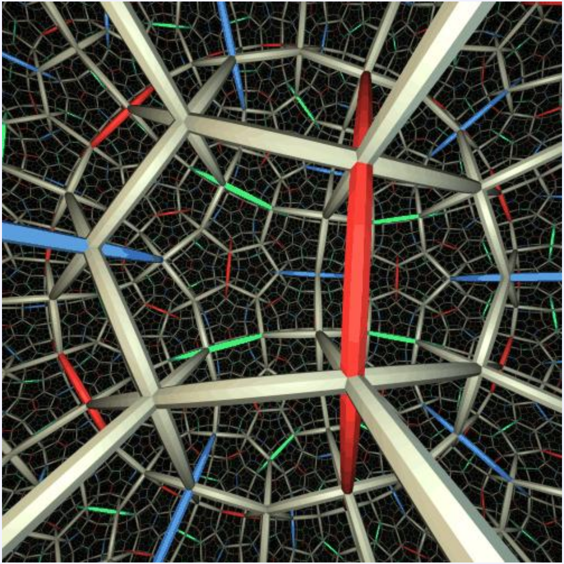
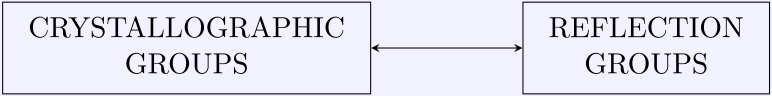


image from W. Thurston's book


CRYSTALLOGRAPHIC GROUPS
REFLECTION GROUPS

Definition

Let $X = \mathbb{H}^n$ be the n-dimensional hyperbolic space.

A **hyperbolic reflection group** Γ is a subgroup of the group of isometries of X generated by reflections and such that:

- the action $\Gamma \times X \to X$ is properly discontinuous;
- the quotient space X/Γ has finite volume.

Remark. A fundamental region of a hyperbolic reflection group can be non-compact.

An example

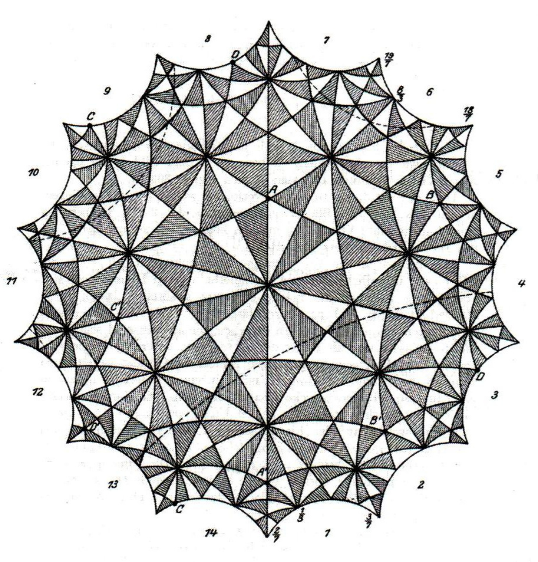


image from F. Klein's paper (1879).

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Theorem 4. [Agol-B.-Storm-Whyte; Nikulin, 2008] *The number of commensurability classes of arithmetic hyperbolic reflection groups in all dimensions is finite.*

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Morally this is the main idea of the proof but unfortunately the actual argument requires more care.

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Question 4. *Do there exist any hyperbolic lattices in the spaces of large dimension which are generated by elements of finite order? Is there a finiteness theorem for such lattices in small dimensions?*

Vinberg's algorithm setup

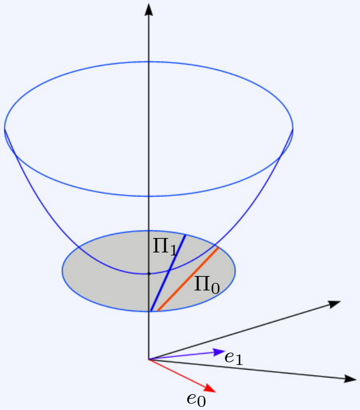
Vector model of $\mathbb{H}^n$: $\mathbb{E}^{n,1}$ with inner product defined by a quadratic form f of signature $(n, 1)$:

$\mathcal{C} \cup -\mathcal{C} = \{v \in \mathbb{E}^{n+1} \mid (v, v) < 0\}$;
$\mathbb{H}^n$ = set of rays through 0 in $\mathcal{C}$.

Hyperplane Π_e — rays in $\mathcal{C}$ orthogonal to $e \in \mathbb{E}^{n,1} \mid (e, e) > 0$.

Reflection in Π_e is

$$R_e : x \to x - 2\frac{(e, x)}{(e, e)}e$$



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- Choose the next vector e_{i+1} such that:
  (a) $2\frac{(e_{i+1}, v_i)}{(e_{i+1}, e_{i+1})} \in \mathcal{O}_k$ for the basis vectors v_i, $i = 1, \ldots, n+1$ *(the crystallographic condition)*;
  (b) $(e_{i+1}, e_{i+1}) > 0$, $(e_{i+1}, u_0) < 0$, and $(e_{i+1}, e_j) \leqslant 0$, $\forall j \leqslant i$;
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  (c) the distance btw. x_0 and $\Pi_{e_{i+1}}$ is the smallest possible.
- Repeat the previous step until get a finite volume polyhedron.

Example for Vinberg's algorithm

$$f = 2\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_1x_2 - \frac{1}{2}(1+\sqrt{5})x_2x_3 - x_3x_4\right).$$

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The next vector is $e_4 = (-1,-1,-\phi,-\phi)$. And the algorithm terminates.

The Coxeter diagram of of the resulting configuration is

Coxeter diagrams

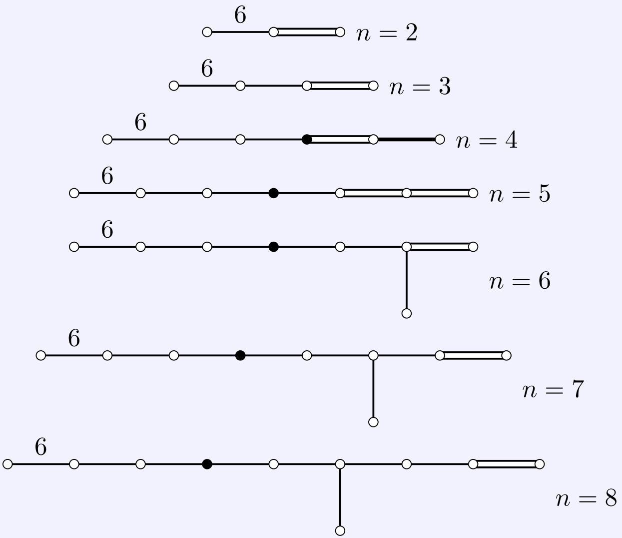
Vertices $\longleftrightarrow$ Faces of the polyhedron

Table: Edges of the diagram

Type of edge	Corresponds to
$m-2$ lines or label m	dihedral angle $\frac{\pi}{m}$
a thick line	a “cusp”, dihedral angle 0
a punctured line	divergent faces
no line	dihedral angle $\frac{\pi}{2}$

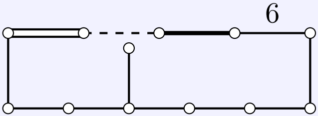
Examples

$f = -3x_0^2 + x_1^2 + \ldots + x_n^2$, for $n = 2$ to 8.



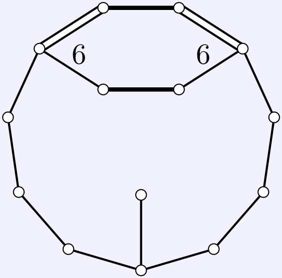
Examples

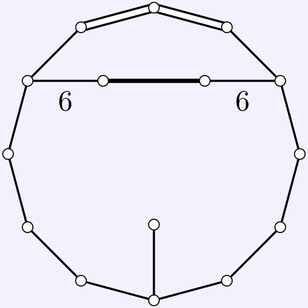
$$f = -3x_0^2 + x_1^2 + \ldots + x_n^2, \;\; n = 9,\ 10,\ 11.$$



$n = 9$

$n = 10$

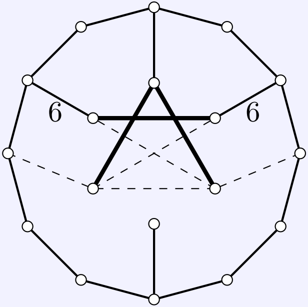




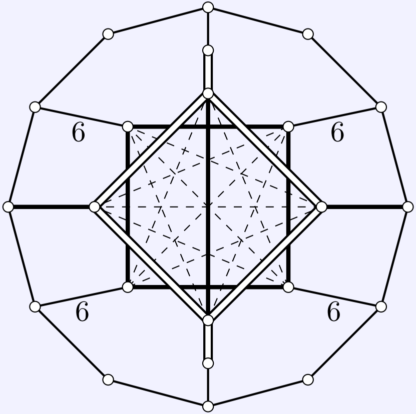
$n = 11$

Examples

$n = 12$:

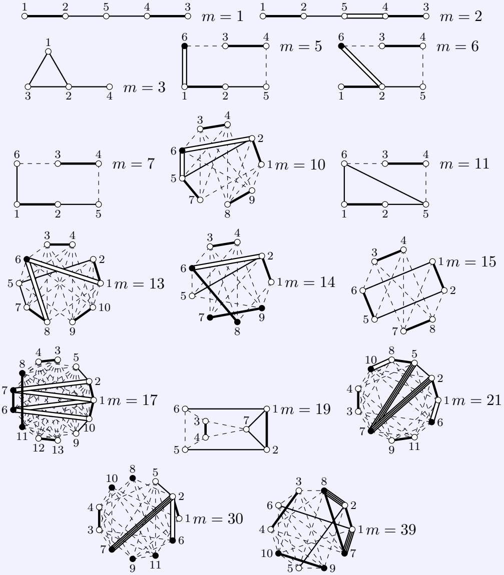


$n = 13$:



J. Mcleod, *Hyperbolic reflection groups associated to the quadratic forms* $-3x_0^2 + x_1^2 + \ldots + x_n^2$, Geom. Dedicata. **152** (2011), 1–16.

Examples



B. – Mcleod, *Reflective and quasi-reflective Bianchi groups*, Transform. Groups, 18 (2013), 971–994.

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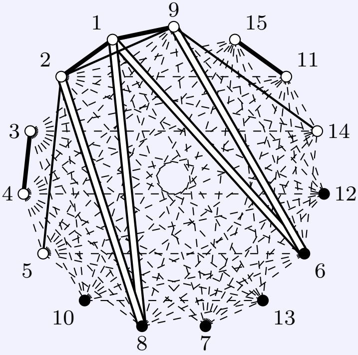

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A reference

M. Belolipetsky, *Arithmetic hyperbolic reflection groups*, survey article, Bull. Amer. Math. Soc. (N.S.) 53 (2016), no. 3, 437–475.