Cells in Coxeter Groups

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Intro

Let W be a Coxeter group with generating set S and defining relations of the form $(st)^{m_{st}} = 1$ for pairs of generators $s, t \in S$. In 1979 paper Kazhdan and Lusztig have defined a partition of W into various classes of subsets called *cells.*

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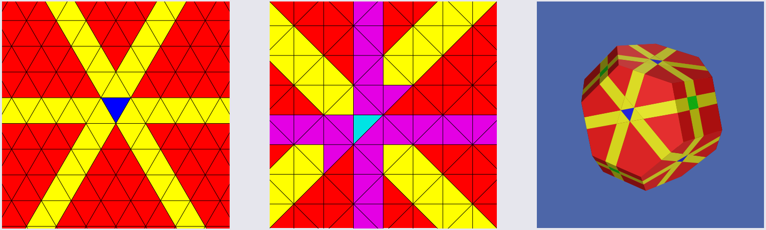
Cells can be visualized via the action of W on its Tits cone:



The cells of $\widetilde{\mathrm{A}}_2$, $\widetilde{\mathrm{C}}_2$, and $\widetilde{\mathrm{A}}_3$

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- representations of p-adic groups (Lusztig, 83);
- characters of finite groups of Lie type (Lusztig, 84);
- the geometry of nilpotent orbits in simple complex Lie algebras (Lusztig, 89; Bezrukavnikov-Ostrik, 04).

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- the geometry of nilpotent orbits in simple complex Lie algebras (Lusztig, 89; Bezrukavnikov-Ostrik, 04).

To illustrate the last we mention an important *result of Lusztig:*
If W is the affine Weyl group attached to the simple complex algebraic group G with Lie algebra $\mathfrak{g}$, then the two-sided cells are in bijection with the set $\mathcal{O}({}^L\mathfrak{g})$ of nilpotent orbits of the group dual to G.

Definitions

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$P_{y,w}$ define preorders $\leq_L$, $\leq_R$, $\leq_{LR}$ and the associated equivalence relations $\sim_L$, $\sim_R$, $\sim_{LR}$ on W. The equivalence classes are called *left cells* (resp. *right cells,* resp. *two-sided cells*).

Definitions

Multiplication:

$$C_x C_y = \sum_z h_{x,y,z} C_z, \quad h_{x,y,z} \in \mathcal{A}$$

$a(z)$ is the smallest integer such that
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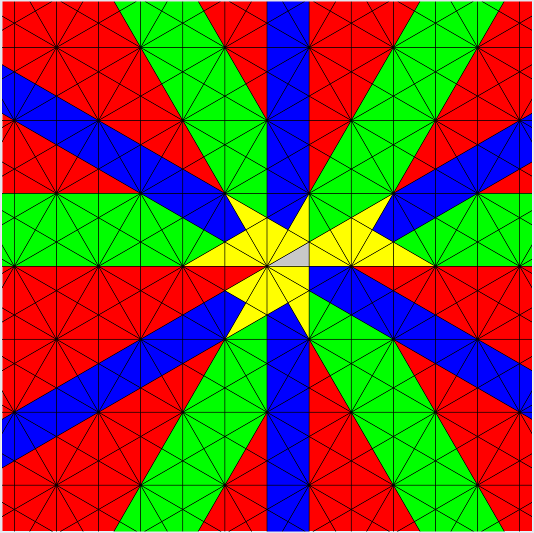
$\mathcal{D}_i := \{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}$, where $\delta(z) = \deg(P_{e,z})$.

The set $\mathcal{D} = \mathcal{D}_0$ consists of *distinguished involutions* of W.

Every left cell of W contains a unique $d \in \mathcal{D}$ (Lusztig, 87).

Pictures



The cells of $\widetilde{\mathrm{G}}_2$ (Lusztig, 85)

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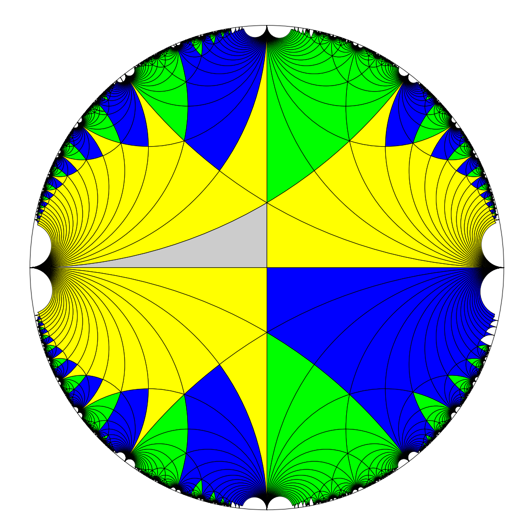

The cells of the modular group $(2, 3, \infty)$ (Gunnells)

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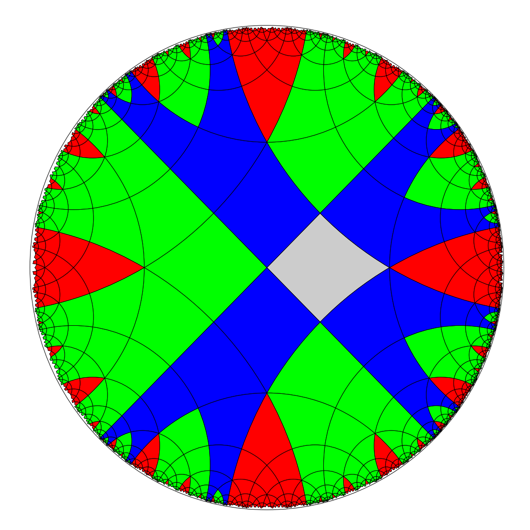

The cells of the group $(2, 2, 2, 3)$ (Gunnells)

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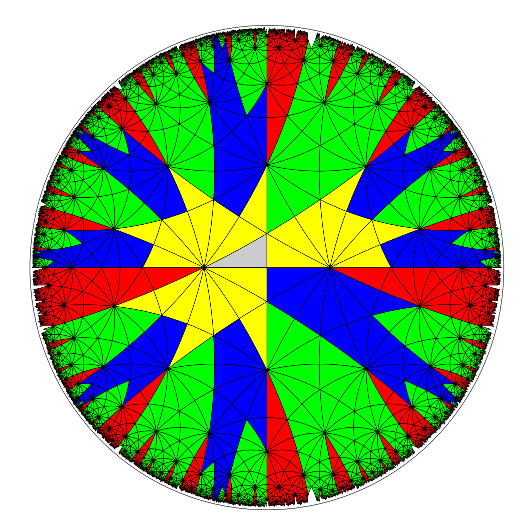
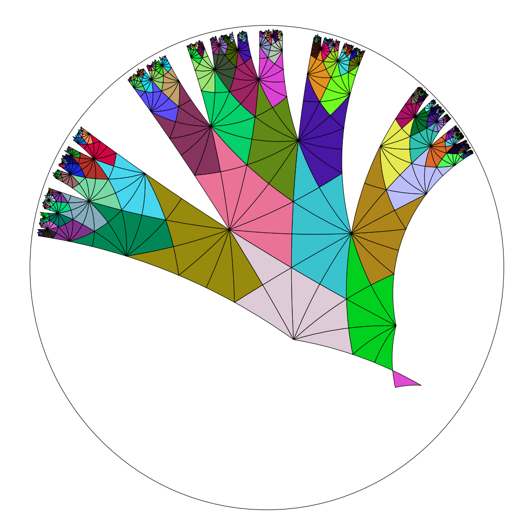


The cells of the Hurwitz group $(2, 3, 7)$ (Gunnells)

Lego game



A left cell (green) of the Hurwitz group

Definitions

Let $w \in W$. $Z(w)$ denotes the set of all $v \in W$ such that $w = x.v.y$ for some $x, y \in W$ and $v \in W_I$ for some $I \subset S$ with W_I finite.

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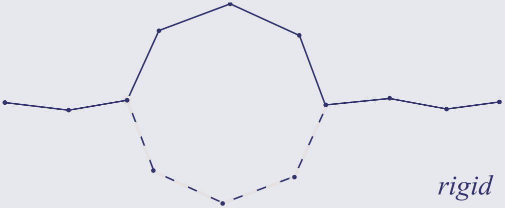

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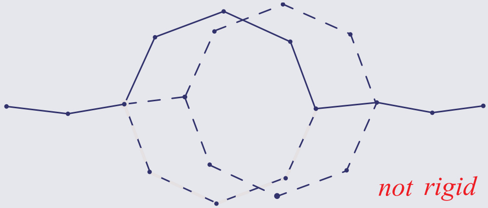

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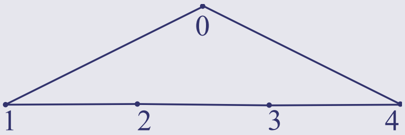
Conjecture 2. ("basic equivalences") *Let $w = x.v_0$ with $v_0 \in \mathcal{D}_f$ maximal in w. Let $u = x.v_1.x^{-1} \in \mathcal{D}$ satisfy $a(u) \leq a(v_0)$, and let $w' = w.u.v_{01}$ where v_{01} is a product of $a(u) - 1$ simple reflections from $\mathcal{R}(v_0)$. Assume $a(w') = a(w)$ and $v_0''xv_1$ is rigid at v_1 for every v_0'' such that $v_0 = v_0'.v_0''$, $l(v_0'') = l(v_{01})$. Then $\mu(w, w') \neq 0$ and $w \sim_R w'$.*

Example

Let W be the affine group of type $\tilde{A}_4$:



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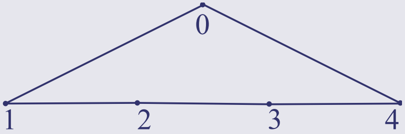

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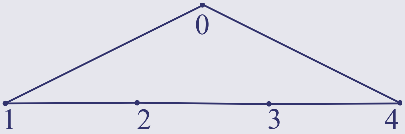

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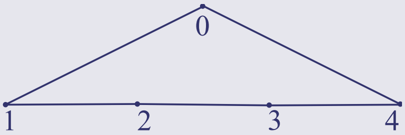

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However, $s_0 s_2 s_3 s_1\ s_4 s_0 s_4 s_2\ s_1 s_3 s_2 s_0 \notin \mathcal{D}$ *(!)*

This is because $s_0 s_2 s_3 s_1\ s_4 s_0 s_4 s_2$ is not rigid at v_1:

$$s_0 s_2 s_3 s_1\ s_4 s_0 s_4 s_2 = s_2 s_0 s_3 s_1\ s_0 s_4 s_0 s_2 = s_2\ s_3 s_0 s_1 s_0\ s_4 s_0 s_2,$$

where $s_3 s_0 s_1 s_0 \in \mathcal{D}_f$ and $a(s_3 s_0 s_1 s_0) = a(v_1)$.

Results

Theorem 1. *(MB-Gunnells) If Conjectures 1, 2, and standard conjectures* are true then*

(1) *The set $\mathcal{D}$ of distinguished involutions consists of the union of $v \in \mathcal{D}_f$ and the elements of W which are obtained from them using Conjecture 1.*

(2) *Relations described in Conjecture 2 determine the partition of W into right cells, i.e. $x \sim_R y$ in W if and only if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ in W such that $\{x_{i-1}, x_i\} = \{v, v'\}$ as in the conjecture for every $i = 1, \ldots, n$.*

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*:
- *Positivity* (Kazhdan-Lusztig'79, Lusztig'85);
- *Function a is given by $a(w) = \max_{v \in Z(w) \cap \mathcal{D}_f} a(v)$* (cf. Lusztig'03).

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Theorem 3. *(MB-Gunnells) Let* $w = x.v_0 = t_n \dots t_1.s_l \dots s_1$ *with* $v_0 = s_l \dots s_1 \in \mathcal{D}_f$ *is maximal in* w *and is the longest element of some* W_I*, and* $a(w) = a(v_0)$*;* $u = y.u_0.y^{-1} \in \mathcal{D}$ *with* $u_0 \in \mathcal{D}_f$ *such that* $a(u) = a(u_0) = l$*; and* $w' = w.u.v_{01}$ *with* $v_{01} = s_1 \dots s_{l-1}$ *has* $a(w') = a(w)$ *and* $\mathcal{R}(w') \subsetneq \mathcal{R}(w)$*. Moreover, assume*

(1) For any $v_j = t_j \dots t_1 v_0 t_1 \dots t_j$, $j = 0, \dots, n-1$ and $t = t_{j+1}$ or $t = t_{j-1}$ if $t_{j-1} \notin \mathcal{R}(v_j)$, we have $a(v_j t) = a(v_j)$, $\mathcal{L}(v_j t) \smallsetminus \mathcal{R}(v_j t) \neq \emptyset$ and $t v_j t$ is rigid at v_0.

(2) For any $u_j = s_{j-1} \dots s_1 u s_1 \dots s_{j-1}$, $j = 1, \dots, l-1$ with $u_1 = u$, we have $a(u_j s_j) = a(u_j)$, $\mathcal{L}(u_j s_j) \smallsetminus \mathcal{R}(u_j s_j) \neq \emptyset$ and $s_j u_j s_j$ is rigid at u_0;

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(a) show that $\delta(vs)(= \deg(P_{e,vs})) = \delta(v)$;

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Proof of Thm. 3 uses Thm. 2 and the equality

$$P_{v_0,\ v_0uv_{01}}=P_{e,\ v_0uv_{01}}$$

(Kazhdan-Lusztig).

Applications

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Application 2. If there exist $s, t_1, t_2 \in S$ such that $m(s, t_1) = m(s, t_2) = \infty$ and $m(t_1, t_2)$ is finite, then W has infinitely many one-sided cells.

Applications

Application 1. Cells in Coxeter groups with with equal exponents, e.g. $\widetilde{\mathrm{A}}_2$, right-angled Coxeter groups (MB'04), etc.

Application 2. If there exist $s, t_1, t_2 \in S$ such that $m(s, t_1) = m(s, t_2) = \infty$ and $m(t_1, t_2)$ is finite, then W has infinitely many one-sided cells.

Rem. Application 2 does not cover the Hurwitz group $(2, 3, 7)$.

Some references

D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

P. Gunnells, Cells in Coxeter groups, *Notices of the AMS* **53** (2006), 528–535.

M. Belolipetsky, Cells and representations of right-angled Coxeter groups, *Selecta Math., N. S.* **10** (2004), 325–339.

M. Belolipetsky, P. Gunnells, Cells in Coxeter groups, *preprint*.