Periods of families of curves in threefolds 16/03/2022 Sheffield

Max

Noether-Lefschetz thm: A generic surface $X \subseteq \mathbb{P}^3$ has Picard rank one.

$\deg f = d$ X $f(x_1, x_2, x_3, x_4) = 0$ $\mathbb{P}^3(\mathbb{C})$

homog. $\mathbb{P}^3$

$$f = \sum t_\alpha x^\alpha \qquad x^\alpha = x_1^{\alpha_1} \cdots x_4^{\alpha_4}$$

$\alpha_1 + \alpha_2 + \cdots + \alpha_4 = d$

$T = (\cdots t_\alpha \cdots) = \mathbb{C}^{\#\text{of monomals of degree } d}$.

$$T - \left(\{f_1 = 0\} \cup \{f_2 = 0\} \cup \cdots \quad \{f_k = 0\} \cdots \right)$$

Picard rank $(X) = 1$. $\equiv$ only curves in X are obtained by intersecting X with another surface.

X: $\underbrace{f(x_1, x_2, \ldots, x_4) = 0 \quad g(x) = 0}_{C}$

$f = \{x_1^d + x_2^d + x_3^d + x_4^d\}$ $\{g = 0\}$

$x_1 - \xi_{2d} x_2 = x_3 - \xi_{2d} x_4 = 0$ $\subseteq X$ $f = 0$

1924 Lefschetz. 1980 Griffiths:

given $X \subseteq \mathbb{P}^3$, it is very difficult to decide wh $\rho(X) = 1$

Hodge cycles $=$

$H^{1,1} \cap H^2(X, \mathbb{Z})$.

$H^2(X, \mathbb{Z})$

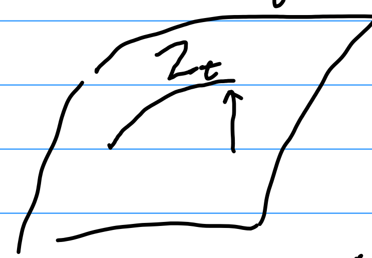
$\cap$

$H^2(X, \mathbb{C}) = H^{20} \oplus H^{11} \oplus H^{02}$

$\rho(X) = \mathrm{rank}_{\mathbb{Z}}(\quad)$

$X \subseteq \mathbb{P}^3$

X_t



Idea of a proof due to Lefschetz, Griffiths, ...

$H^{2,0}$ - space of $\boxed{\text{hol 2-forms}}$ in X

$\omega \in H^{2,0}$

$\omega\big|_{Z_t} \equiv 0$ $\quad d \geqslant 4.$

Z_t $\quad$ 1

$t \in T = \mathbb{C}^{\#}\ldots$

$[Z_t] \in H_2(X_t, \mathbb{Z})$
$\quad\ \ \| $
$\quad\ \delta_t \, \mathbb{C}$

$\Rightarrow \int_{[Z_t]} \omega = 0$ $\quad t$

hol fcn. in t

derivating integrals which dep on parameters $\equiv$ Gauss-Manin Connection

$$\frac{\partial}{\partial t_\alpha} \int_{\delta_t} \omega = \int_{\delta_t} \frac{\partial \omega}{\partial t_\alpha}$$

$$\int_{[Z_t]} \text{primitive part of } H^2_{dR}(X) \equiv 0 \overset{?}{\Longrightarrow} [Z_t] \text{ is } X \cap Y_t. \blacksquare$$

Codim 1

$\deg f = 5$

Quintic threefolds in $\mathbb{P}^4$: $X \subseteq \mathbb{P}^4$ $\quad f(x_1, x_2, \dots, x_5) = 0$ homog.

$/\mathbb{C}$

Rational curves: $\alpha : \mathbb{P}^1 \longrightarrow \mathbb{P}^4$

$[x:y] \longmapsto [f_1(x,y) : f_2(x,y) : \cdots f_5(x,y)]$

$\mathrm{Im}(\alpha) \subseteq X$ $\quad f_1, f_2 \dots f_5$ homog. poly in x,y of degree d.

$\rightsquigarrow X$ has a rational curve of degree d.

Clemens conjecture: ~~for a~~ generic X, has a finite number of rational curvs in each degree $\underline{d}$.

a rational curve of degree d in X

$$f(f_1(x,y), f_2(x,y), \dots, f_5(x,y)) = 0.$$

the number of $\underbrace{\text{equations}}_{\text{linear}}$ = the number of coef of $f_1, f_2 \dots f_5$

S. Katz

$d = 1$

2875 lines inside a generic quintic $\quad \mathbb{P}^1 \subseteq X$

$d \leq 11$ ✓

A-r

1991 String th. Candelas, de - - -

$$\underline{\underline{Y}} = 5 + 2875 \frac{q}{1-q} + 609250 \cdot 2^3 \frac{q^2}{1-q^2}$$

$$+ \dots \quad n_d \, d^3 \frac{q^d}{1-q^d}$$

n_d = # rational curvs of degree d in a generic quintic -thr

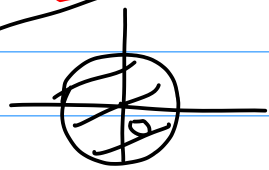
Givental, Kontse —

virtual number of rationals.

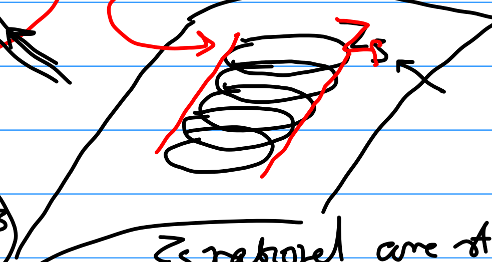
Thm (M.): If Clemens conjecture is false in degree d, then for a hol. family of degree d curves $Z_s, s\in(\mathbb{C},0)$ in a generic quintic threefold X and $\omega\in F^2H^3_{dR}(X)$ we have $\int_{Z_s}\frac{\omega}{ds}\equiv 0.$

ω X

3



$\mathbb{C}$
$\cup$
S



Z_s

$H^3_{dR}(X)\simeq H^3(X,\mathbb{C})=H^{30}\oplus H^{21}\oplus H^{12}\oplus H^{03}$

Z_s rational curve of degree d.

harmonic forms.

(p,q)-forms

$(z_1,z_2,\dots z_n)$ local.

$dz_1, dz_2 \dots dz_n \quad d\bar{z}_1\ d\bar{z}_2 \dots d\bar{z}_n$

C^∞

$F^3\ \underbrace{H^{30}\oplus H^{21}}\oplus \dots$

Grothendieck

Hodge filtration Deligne

$F^2 = F^2H^3_{dR}(X)$

C^∞

3

$\frac{\omega}{ds}$ Gelfand-Leray form

1

2-forms.

$$\int_{[Z_s]}\frac{\omega}{ds}\equiv 0.$$

$[Z_s]\in H_2(X_t,\mathbb{Z})$

Given $\omega \in F^2 H^3_{dR}(X)$ and

family of curves Z_s, $s \in \mathbb{P}^1$. it is possible to compute

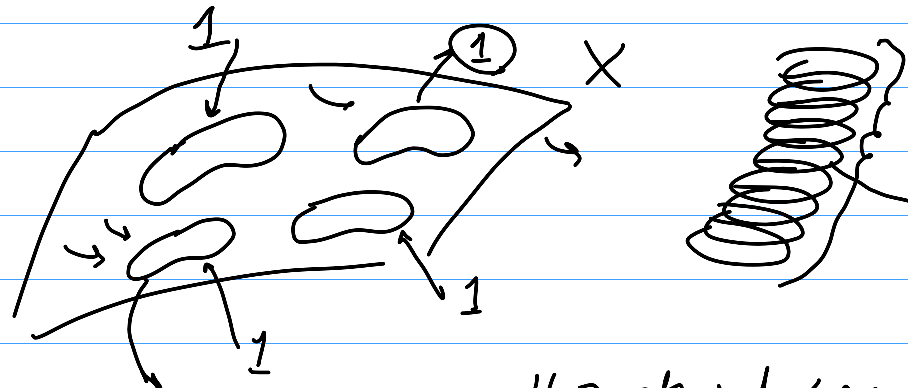
$$\boxed{\frac{1}{2\pi i}\int_{Z_s} \frac{\omega}{ds}} \in \overline{\mathbb{Q}(s)}$$

$= 0$ or not

Residue calc.

1. generic X $\quad \int \circ = 0.$

2. $\int_{[Z_t]} \text{primi}\left(H^2_{dR}(X)\right) \equiv 0 \overset{?}{\Longrightarrow} Z_t = X \cap Y_t.$



$\gamma \in \mathbb{Q}$

$\sum \gamma$

rational curve degree d

n_d

$\mathbb{Z}/N$.

$d \ll 11$ $\qquad$ $n_d \in \mathbb{Q}$.