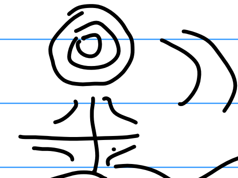
Space of hol. fol. with center / $x^2+y^2=(x+iy)(x-iy)$

Center $\quad x^2+y^2=\text{const}$  $\mathbb{C}$

$\quad xy=\text{const}$

$\mathcal{F}(d) :=$ foliations $\mathcal{F}(\omega)$, $\omega = P(x,y)\,dy - Q(x,y)\,dx$

deg $P, Q \le d$

$\mathcal{M}(d) :=$ foliations with a center $\quad \omega \wedge d(\tilde{x}^2+\tilde{y}^2)=0$

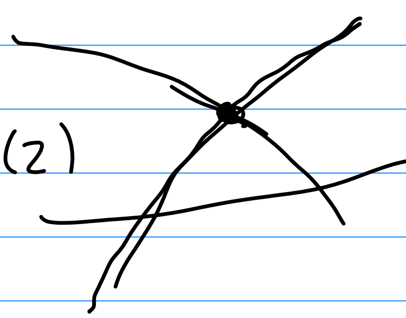
formal. $\quad \underset{\mathbb{C}^2}{(x,y)}$

$\omega \wedge d(x^2+y^2+\text{h.o.t})=0$

$\longrightarrow$ algebraic expr. for coef.s of P, Q

$\Rightarrow \mathcal{M}(d)$ is an algebraic subset of $\mathcal{F}(d)$

Classify components of $\mathcal{M}(d)$



Dulac 1908: gives all components of $\mathcal{M}(2)$

$d=3$

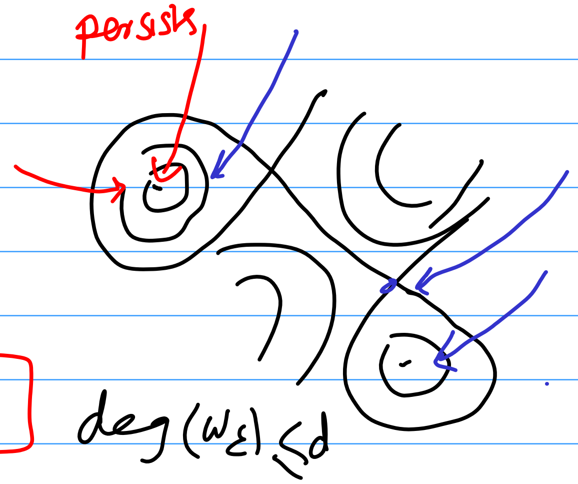
Cerveaux–Lins Neto: $\underset{\text{proj.}}{\mathbb{P}^2} \supseteq \mathbb{C}^2 \leftarrow$ affine.

$\Rightarrow$ if $\mathcal{F}$ in $\mathbb{P}^2$ of degree 2 has center then it must have invariant line.

Ilyashenko 1969: Hamiltonian d.e.

$$\{df \mid f \in \mathbb{C}[x,y]_{\le d+1}\}$$

form an irr. compon. of $\mathcal{M}(d)$.



$f=\text{const}$

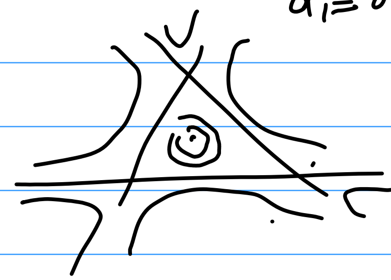
$$\omega_\varepsilon = \boxed{df + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots} \qquad \deg(\omega_\varepsilon) \le d$$

$$\Longrightarrow \underline{\underline{\omega_\varepsilon = df_\varepsilon}} \qquad f_\varepsilon \in \mathbb{C}[x,y] \qquad \mathbb{C}^2$$

logarithmic foliations $\boxed{\omega = f_1 f_2 \cdots f_s \left(\sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}\right)}$

$f_i \in \mathbb{C}[x,y]_{\le d_i}$ $\quad \deg(\omega)=d$

$d_1+d_2+\cdots d_s - 1 = d$ $\quad d_1=d_2=d_3=1$

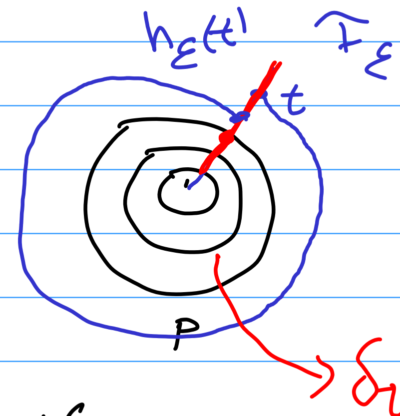


M. 2004: $\mathcal{L}(d_1,d_2,\ldots d_s)$ is a irr. component of $\mathcal{M}(d)$.

Z. 2019: pull-back dif equ.

Idea of the proof of Ilyashenko's thm

$\mathcal{F}_\varepsilon$ $\quad df+\varepsilon\omega+\cdots$, $\quad f\in\mathbb{C}[x,y]_{\le d}$ $\quad \deg(\omega)\le d$



$$h_\varepsilon(t) = t + \varepsilon M_1(t) + \varepsilon^2 M_2(t) + \cdots$$

$$\boxed{M_1(t) := \int_{\delta_t} \omega_1 \equiv 0}$$

df $\quad \delta_t \subseteq$ fibers of f

$$M_1 = M_2 = \cdots \equiv 0$$

Picard-Lefschetz thm. action of monodromy

$$\int_{H_1(\text{fibers of } f)} \omega_1 \overset{0}{\equiv}$$

$\Downarrow$ $\quad \deg(\omega_1) \le d$

$$\omega_1 = df_1 \qquad f_1 \in \mathbb{C}[x,y]_{\le d+1}$$

$$T_{\mathcal{F}_0}\mathcal{M}(d) = \underline{\underline{T_{\mathcal{F}_0}\mathcal{H}(d)}} \Longrightarrow$$

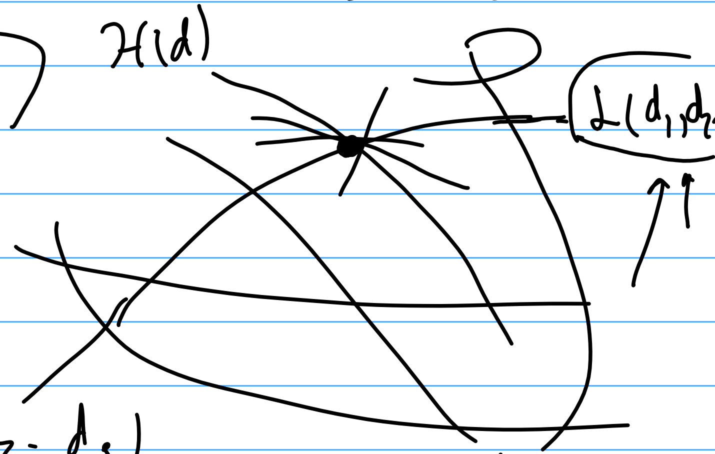
$\mathcal{F}_0 = \mathcal{F}(df)$

$\boxed{f: \text{ has } d^2 \text{ non deg. critical points with distinct values}}$

$\hookrightarrow$ [$f = f_0 \cdots f_{d+1}$ $\quad f_{d+1} = 0$ is square free $\quad \mathcal{H}(d).$]

generic log. foliation: $\quad \mathcal{L}(d_1, d_2 \ldots d_s).$

there is no P.L. theory for



$f = l_1 l_2 \cdots l_{d+1}$ $\quad l_i$ deg 1.

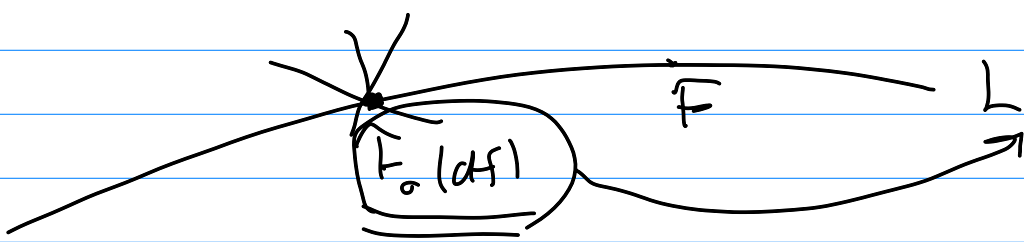
$\mathcal{L}(d_1, d_2 \ldots d_s)$

$$\frac{df}{f} = \underbrace{\sum \frac{dl_i}{l_i}} \Rightarrow \text{logarithmic foliation}$$

$$\underbrace{TC_{\mathcal{F}_0(df)} M(d)} = \bigcup TC(\mathcal{L}(d_1 \ldots d_s))$$

2004

all logarithmic comp. $\mathcal{L}(d_1 \ldots d_s)$ passing through $\mathcal{F}_0(df)$



$\mathcal{L}(d_1, d_2 \ldots d_s)$
$\cup I$
$M(d)$

$T_{\mathcal{F}} M(d) = T_{\mathcal{F}} \mathcal{L}(d_1, \ldots d_s)$ $\quad$ generic $\mathcal{F} \in \mathcal{L}(d_1 \ldots)$

$\mathcal{F}: \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i}$ $\quad$ The only deformations of $\mathcal{F}$ with center is ~~the one~~ by def for λ_i, f_i

$$\mathcal{L}(\underbrace{1,1,1,\dots 1}_{d+1}) = \mathcal{L}(1^{d+1})$$

$\deg f_i = 1$, l_i $: \sum \lambda_i \frac{dl_i}{l_i}$, $n_i \in \mathbb{N}$, $\lambda_i = n_i \in \mathbb{N}$

$$f = l_1^{n_1} l_2^{n_2} \cdots l_{d+1}^{n_{d+1}}$$

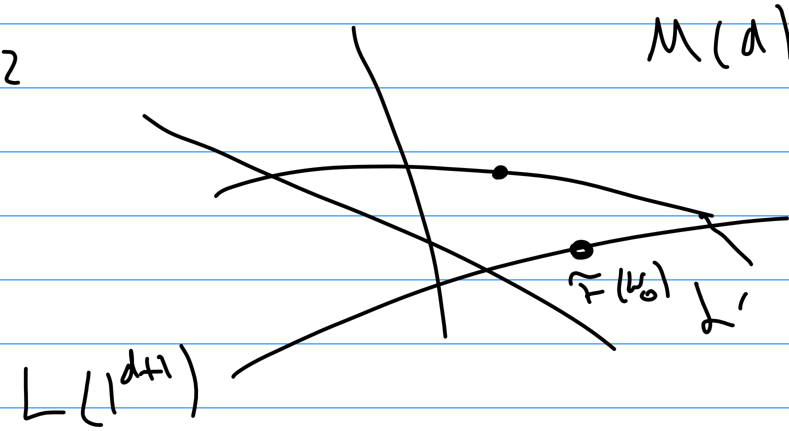
$$\omega_0 = l_1 l_2 \cdots l_{d+1} \left(\sum_{i=1}^{d+1} n_i \frac{dl_i}{l_i} \right) \quad \mathcal{F}(\omega_0).$$

Q: Is $\mathcal{F}(\omega_0)$ a smooth point of $M(d)$?

Answer: No if $n_1 = n_2 \to f_2$

$$n_1 \frac{dl_1}{l_2} + n_1 \frac{dl_2}{l_2} = n_1 \left(\frac{d l_1 l_2}{l_1 l_2} \right).$$

$\mathcal{F}(\omega_0) \in \mathcal{L}(1,\dots,2,\dots)$.



$n_i \neq n_j$

Thm (M. Gavrilov): $n_i \in \mathbb{N}$, n_i's pairwise coprime distinct.

$\Rightarrow \mathcal{F}(\omega_0)$ is a smooth points of $M(d)$.

$$T_{\mathcal{F}(\omega_0)} M(d) \supseteq T_{\mathcal{F}(\omega_0)} \mathcal{L}(1^{d+1}).$$

trivial

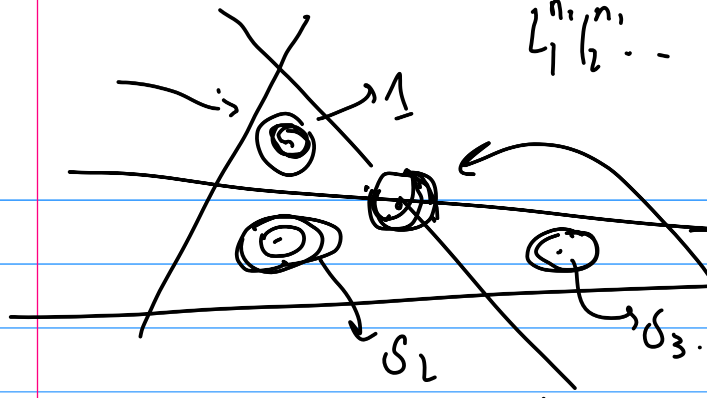
?

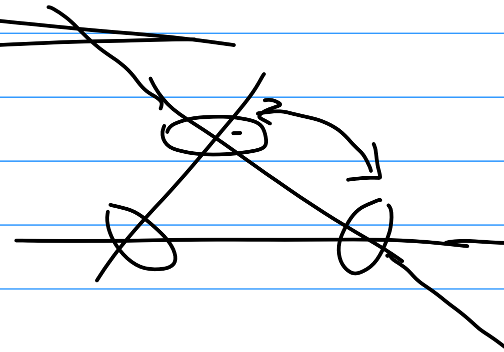
$$f = l_1^{n_1} l_2^{m_2} \cdots l_{d+1}^{n_{d+1}} \quad \subseteq \quad (x^n y^m = \text{const}).$$

Clement $f = 0$.

$l_1^{n_1} l_2^{n_1} \cdots = \text{const}$

$\underbrace{\int_{\delta_1} w = 0}$ $\Longrightarrow$ $\int_{\delta_2, \delta_3} w = 0$





$(n_1, n_2) \neq 1$

saddle conn.

$$g\left(\overline{\text{fibes of } f}\right) = \frac{1}{2}\left((d-1)n - \sum_{i=1}^{d+1} (n_i, n)\right)$$

$$\underline{\underline{n = \sum n_i}}.$$