Today I met Raphael and he told me 24/09/2021

$$H^1(M,\mathbb{Z}) \cong \{f: M \xrightarrow{C^\infty} S^1\} / \text{Homotopic maps.}$$

If $\delta \simeq S^1 \in H_1(M,\mathbb{Z})$ then $\int_\delta f: M \to S^1 :=$ the degree of $f: \delta \to S^1$.

Question: Can we generalize this to higher cohomologies

Obs1: Any element in $H_1(M,\mathbb{Z})$ is a $\mathbb{Z}$-linear sum of cycles difeomorphic to S^1. In general this might not be true.

Obs2: The above statement is true for smooth hyps $X \subseteq \mathbb{P}^{n+1}(\mathbb{C})$ of odd dimension (n odd)

$H_n(X,\mathbb{Z})$ is generated by "vanishing cycles" $S^n \subseteq X$.

But for n even

$H_n(X,\mathbb{Z})$ is generated by vanishing cycles + polarization which is

$$[\mathbb{P}^{\frac{n}{2}+1} \cap X] \subseteq H_n(X,\mathbb{Z})$$

see chapters 5,6,7 of my Book "A course in Hodge The" with emphasis on multiple.

Any way Picard-Lefschetz theory and vanishing cycles tell us that

$$\{f: M \xrightarrow{C^\infty} S^n\} / \text{homotopic maps.}$$

might be very close to $H^n(X,\mathbb{Z})$, at least in the case of smooth hypersurfaces. $X \subseteq \mathbb{P}^{n+1}(X)$.