30/03/2020. Rio de Janeiro

Let $F(x_1, \ldots x_{n+1})$ be a homogeneous polynomial in $x_1, \ldots x_{n+1}$ and assume that $\mathbb{P}\{F=0\} \subseteq \mathbb{P}^n$ is smooth. Equivalently

$$\dim_{\mathbb{C}} S < \infty \quad \text{where} \quad S = \frac{\mathbb{C}[x]}{\mathrm{Jacob}(F)}$$

Let also $T := \mathbb{C}^N \setminus \{\Delta = 0\}$ be the parameter space of such poly's.

Thm: We have

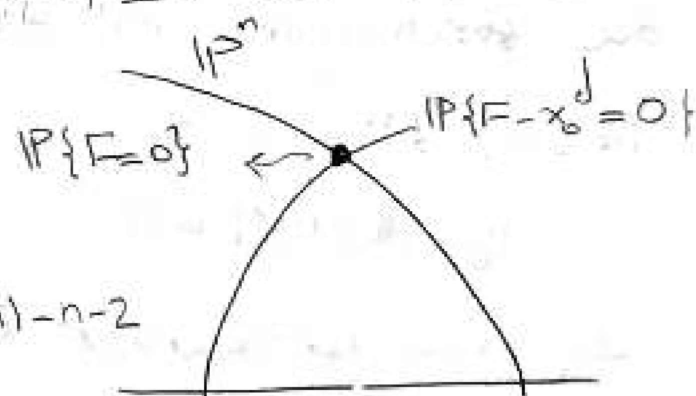
$$\dim S_{dk-(n+1)} = h^{n-k,k+1}\left(\mathbb{P}\{F=0\} \subseteq \mathbb{P}^n\right)$$

$$\dim \bigoplus_{kd-(n+1) < a \leqslant (k+1)d-(n+1)} S_a = h^{n-k,n}\left(\mathbb{P}\{F - x_0^d\} \subseteq \mathbb{P}^{n+1}\right)$$

and so these are constant functions on T.

Proof: Both follow from the Griffiths description of the de Rham cohomology of $\mathbb{P}\{F=0\} \subseteq \mathbb{P}^n$ and $\mathbb{P}\{F - x_0^d = 0\} \subseteq \mathbb{P}^{n+1}$

For the second part we may a little bit argue



$$\left(\mathbb{C}[x_0, x] / \mathrm{Jacob}(F - x_0^d)\right)_{d(k+1)-n-2}$$

$$\left(\mathbb{C}[x_0, x] / \mathrm{Jacob}(F) + \langle x_0^{d-1} \rangle\right)_{d(k+1)-n-2}$$

$$\simeq \left(\mathbb{C}[x] / \mathrm{Jacob}(F)\right)_{d(k+1)-n-2} \oplus \left(\mathbb{C}[x] / J(F)\right)_{d(k+1)-n-3} x_0$$

$$\oplus \cdots \left(\mathbb{C}[x] / J(F)\right)_{d(k+1)-n-2-(d-2)} x_0^{d-2}$$

which finishes the proof ∎

How about $\dim S_i$, $i \neq dk-(n+1)$?

Obs:1 $\dim S_i$ are descrete numbers and so in a Zariski open set $\dim S_i$ is constant for all i.

The interpretation of $\dim S_i$: $kd-(n+1) < i < (k+1)d-(n+1)$

For $P \in S_i$ we have

$$\operatorname{Res}\left(\frac{P \sum_{i=0}^{n+1} (-1)^{i-1} x_i \, d\hat{x}_i}{(F-x_0^d)^{k+1}} \right) \in H^{n-k,n}(F - x_0^d = 0).$$

If we restrict this diff. form to $x_0 = 1$ then we can compute the residue. Up to multiplication by a constant it is

$$\frac{P(1,x)\, dx_1 \wedge dx_2 \wedge \dots dx_{n+1}}{dF}$$

$$= P(1,x) \left(\sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_i \right) \quad (*)$$

see Section 11.4 and Proposition 11.4 of my Hodge Theory Book

These are differential forms in $F=1$ with poles alonge infinety $\mathbb{P}\{F=0\} \subseteq \mathbb{P}^n$.

Conjecture: For the Zariski open set in Obs 1 we have

$$S_i \simeq \left\{ \omega \in H^n_{dR}(F=1) \;\middle|\; \begin{array}{l} \omega \text{ has a pole order} \\ i + 2(n+1) - 1 \\ \text{at } \mathbb{P}(\{F=0\}) \end{array} \right\}$$

Note that the pole order of $(*)$ at ∞ and as a differential form in $\mathbb{C}^{n+1}$ is $\deg P + 2(n+1) - 1$. When we restrict it to $\{F=1\}$ the pole order at ∞ might decrease and in the above conjecture we claim that in some Zariski open subset of the parameter space T, such pole decrease does not happen.

For X a smooth projective variety and Y a smooth/transversal hyperplane section we have the pole order filtration in $X \setminus Y$

$$F_i := \left\{ \omega \in H^n_{dR}(X-Y) \;\middle|\; \begin{array}{l}\omega \text{ is represented by a} \\ \text{diff. form of pole order} \leq i\end{array} \right\}$$

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

Ex1: $X = \mathbb{P}^n$, $Y =$ hyp. this is the Hodge filtration and so the dimension of F_i are constants.

Ex2: $X = \mathbb{P}\{F - x_0^d = 0\} \subseteq \mathbb{P}^{n+1}$, $Y = \mathbb{P}\{F = 0\} \subseteq \mathbb{P}^n$ as in the previous discussion. The dimensions of F_i's in general doesn't seem to be constant.

Obs2: This is the first problem in Hodge Theory I started to think about. It was at MPIM and together with S. Archana (former student of M. Green).

In this paragraph, I would like to analyze the Zariski open set in Obs1:

Let us take $F_c = \sum_{\deg(x^\alpha)=d} c_\alpha x^\alpha$, where c_α's are transcendental numbers algebraically independent over $\overline{\mathbb{Q}}$.

we take a basis P_i. $i=1,\dots,\mu$ of $\mathbb{C}[x]/J(F_c)$.

Prop: There is a Zariski open subset U of T containing $c \in T$ such that for all $t \in U$, $\{P_i, i=1,2,\dots,\mu\}$ is also a basis of $\mathbb{C}[x]/J(F_t)$.

Proof:
$$\begin{bmatrix} x_i P_1 \\ x_i P_2 \\ \vdots \\ x_i P_\mu \end{bmatrix} = \begin{bmatrix} & A_{ij} & \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_\mu \end{bmatrix} + \begin{bmatrix} P_{ij}(x) \end{bmatrix} \begin{bmatrix} \frac{\partial F_c}{\partial x_1} \\ \frac{\partial F_c}{\partial x_2} \\ \vdots \\ \frac{\partial F_c}{\partial x_{n+1}} \end{bmatrix}$$

This is an equality of polynomials and c_α's are alg independent over $\overline{\mathbb{Q}}$. We conclude that A_{ij}'s and the coef.s of P_{ij}'s are rational functions in (c_α) and so it makes sense to substitute c_α with an arbitrary parameter t_α. The Zariski open set U is T minus the zero set of all denominators in A_{ij} and coef.s of P_{ij}. $\blacksquare$

As a corollary, $\dim S_i$ is constant in the open set U of the previous proposition.

Conjecture: The open set U contains the Fermat and so

$$\dim S_i = \# \left\{ x_1^{\beta_1} \cdots x_{n+1}^{\beta_{n+1}} \;\middle|\; \begin{array}{l} 0 \leqslant \beta_i \leqslant d-2 \\ i=1,2,\dots n+1 \\ \beta_1+\beta_2+\cdots \beta_{n+1} = i \end{array} \right\}.$$

I believe that Prop. 10.7 + Prop 11.8 (Hodge Theory Book) can give us some irreducible components of the complement of U in T.