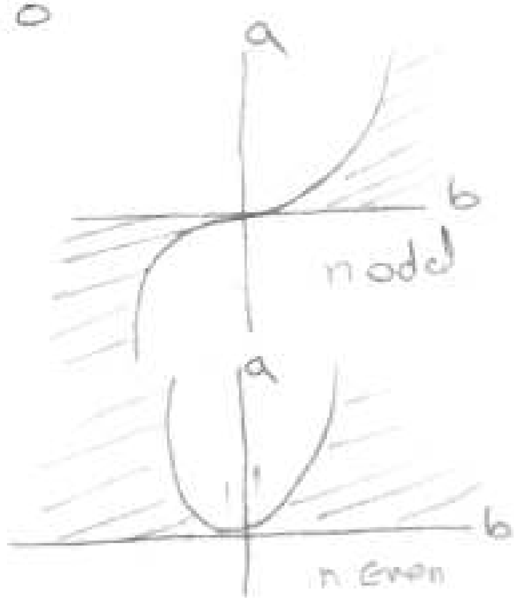
$X(\mathbb{R}) = \{ x \in \mathbb{P}^{n+1}(\mathbb{R}) \mid a x_0^{n+2} + \cdots \quad x_{n+1}^{n+2} - b\, x_0 x_1 \cdots x_{n+1} = 0 \}$

The discriminant $= \Delta = a(a - b^{n+2}) = 0$



There are four different topological type. We take the affine chart $x_0 = 1$

1. In the white region of (a,b), we have the line $b = 0$ and the equation is

$$a + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} = 0$$

n odd : This has always one unbounded component

n even, $a<0$: one bounded component $\simeq_{C^\infty}$ Sphere

n even, $a>0$: empty set

2. For (a,b) in the shade region, that is,

n odd $\quad (a>0 \ \& \ a < b^{n+2})$ or $(a<0 \ \& \ a > b^{n+2})$

n even $\quad (b>0 \ \& \ a < b^{n+2})$ or $(b<0 \ \& \ a > b^{n+2})$

we claim that $X(\mathbb{R})$ has two connected components: one is isomorphic to a sphere in $\mathbb{R}^{n+1}$ and the other intersecting the infinity $\mathbb{RP}(n)$. (unbounded component.

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$f(x_1, x_2, \ldots x_{n+1}) = -x_1^{n+2} - x_2^{n+2} - \cdots - x_{n+1}^{n+2} + b(n+2) x_1 \cdots x_{n+1}$$

$n = 1$

$n = 2$ Do it by yourself in Matlab or mathematica.

$z_1^3 + z_2^3 - b z_1 z_2 = -a$

$(a,b) = (1,2)$

using kmplot in Ubuntu

Proof: $x_i^{n+1} - b\, x_1 \cdots \hat{x}_i \cdots x_{n+1} = 0 \quad i = 1,2,\dots n$

$\Rightarrow \quad x = (b, b, \dots, b)$

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{(b,b,\dots b)} = (-1)\begin{bmatrix} (n+2)(n+1)b^n & & (n+2)b^n \\ & \ddots & \\ (n+2)b^n & & (n+2)(n+1)b^n \end{bmatrix}$$

$$\simeq \begin{bmatrix} (n+1) & & & \\ & (n+1) & 1 & \\ & 1 & & \\ & & & (n+1) \end{bmatrix}_{(n+1)\times(n+1)}$$

This is a positive definite matrix. ∎

The critical value for $(b,b,\dots)$ is $-b^{n+2}$. Therefore, we can see the vanishing cycle of the conifold in $\mathbb{P}^{n+1}(\mathbb{R})$.

An strange observation: For n even a vanishing cycle is present in $\mathbb{P}^{n+1}(\mathbb{R})$ in both side of the path γ in the picture A