Sets of Integers whose subsets have distinct sums

Seminar - The Contributions of John H. Conway

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Introduction

Question: Paul Erdös and Leo Moser asked - Given a positive integer x, what is the maximum number m of positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq x$$

such that all the 2^m possible sums of the a_i:

$$a_{i_1} + a_{i_2} + \cdots + a_{i_j}, \quad 0 \leq j \leq m$$

are different.

- m depends on x, i.e. m is a function of x i.e. $m = f(x)$
- we include 0 as the empty sum in the above
- There are 2^m sums because that is the number of subsets of $\{a_1, \ldots, a_m\}$

Introduction

Equivalent formulation: Given a positive integer x, what is the maximum number m of positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq x$$

such that the sum of the elements of each subset of $\{a_1, \ldots, a_m\}$ is distinct.

Consider when $x = 2^k$

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

Consider the case when $x = 2^k$ in our original question.

Proposition: The set of integers

$$\{2^i \mid 0 \leq i \leq k\}$$

, of cardinality $k+1$, has the property that the sums of all its 2^{k+1} subsets are distinct.

Thus in the case when $x = 2^k$, we see that $k + 1 \leq m$.

Introduction

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

We saw from the proposition on the prev. slide that when $x = 2^k$, we have that $k + 1 \leq m$.

Conjecture: When $x = 2^k$, we must have $m = k + O(1)$. This conjecture is still open. Erdös offers a \$500 reward for the proof or disproof of this.

Introduction

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

Conjecture: When $x = 2^k$, we must have $m = k + O(1)$.

Main goal of the seminar: We saw that when $x = 2^k$, we have that $k + 1 \leq m$. In the case for $x = 2^k$, we will show in this seminar that it is *possible* to have $m = k + 2$.

Remark: In particular this shows that $m \geq k + 2$, but it doesn't go so far as to show that $m = k + 2$ in general for $x = 2^k$.

How to achieve our main goal

How do we achieve this goal? Need to find positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq 2^k$$

such that all possible sums of the a_i are distinct.

How will we find such a_i? Modify the *Conway-Guy* sequence.

Further goals for the seminar: Later on in the seminar we will discuss some results which could be used to resolve this conjecture for arbitrary k.

The case when x is arbitrary

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

- Can find $m = k + 2$ such positive integers a_i when $x = 2^k$ (shall see later)
- Is this maximum such m? What if $x \neq 2^k$?
- Are there any bounds on m?

Answer:

$$\lfloor \log_2 x \rfloor + 1 \leq m < \log x + \frac{1}{2} \log \log x + 1.3$$

where log here means log to the base 2.

First goal of the seminar: Prove the inequality above in the next section.

Goals for the seminar

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

- **Goal 1:** Prove $\lfloor \log_2 x \rfloor + 1 \leq m < \log x + \frac{1}{2} \log \log x + 1.3$
- **Goal 2:** When $x = 2^k$, show that it is possible to have $m = k + 2$.
- **Goal 3:** Prove further properties about the Conway-Guy sequence.

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Introduction

Proposition: The set of integers $\{2^i \mid 0 \leq i \leq k\}$, of cardinality $k+1$, has the property that the sums of all its 2^{k+1} subsets are distinct.

Proof:

- Suppose we have subsets $A = \{2^{i_1}, \ldots, 2^{i_n}\}$ and $B = \{2^{j_1}, \ldots, 2^{j_m}\}$ of $\{2^i \mid 0 \leq i \leq k\}$ such that
$$\sum_{v=1}^{n} 2^{i_v} = \sum_{v=1}^{m} 2^{j_v}$$
- We will show that $A = B$ which will conclude the proof.
- WLOG we can assume that both A and B don't contain $2^0 = 1$ since we can just remove it from both sets in that case to end up with sets $A' = A \setminus \{2^0\}$ and $B' = B \setminus \{2^0\}$ whose elements still sum up to the same value.

Introduction

Proof contd.:

- Recall $A = \{2^{i_1}, \ldots, 2^{i_n}\}$ and $B = \{2^{j_1}, \ldots, 2^{j_m}\}$
- Let $p = \min\{i_1, \ldots, i_n, j_1, \ldots, j_m\}$.
- Assume WLOG that $p = i_v$ for some $1 \leq v \leq n$, then we have that

$$\frac{1}{2^p}\sum_{v=1}^{n} 2^{i_v} = \frac{1}{2^p}\sum_{v=1}^{m} 2^{j_v}.$$

- This implies that

$$1 + \sum_{v=1, v \neq p}^{n} 2^{i_v - p} = \sum_{v=1}^{m} 2^{j_v - p}$$

- Now the left hand side above is odd, and the right hand side is odd if and only if there is a $1 \leq w \leq m$ such that $j_w = p$. If there is no such j_w we arrive at a contradiction.
- Then we can form $A' = A \setminus \{2^p\}$ and $B' = B \setminus \{2^p\}$ and then repeat this process again.
- The end result of this inductive process is that $A = B$ $\square$.

Lower bound for m

Recall our original question.

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

Lower bound for m. Assume we are given some positive integer x. If we set

$$k = \lfloor \log_2 x \rfloor,$$

then the set of integers $\{2^i \mid 0 \leq i \leq k\}$,

- has cardinality $k+1$
- the property that the sums of all its 2^{k+1} subsets are distinct **and**
- $0 < 2^i \leq x$ for all i.

So we have

$$\lfloor \log_2 x \rfloor + 1 \leq m$$

Theorem 1

Theorem 1: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$mx > \sum_{i=1}^{m} a_i \geq 2^m - 1.$$

We obtain equality $\sum_{i=1}^{m} a_i = 2^m - 1$ if $a_i = 2^{i-1}$ for each i.

Proof:

- We first check that $\sum_{i=1}^{m} a_i = 2^m - 1$ if $a_i = 2^{i-1}$ for each i.
- Note that if $a_i = 2^{i-1}$ for $1 \leq i \leq m$, then $a_1, \ldots, a_m$ is a geometric sequence and

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} 2^{i-1} = \frac{1-2^m}{1-2} = 2^m - 1$$

Theorem 1

Theorem 1: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$mx > \sum_{i=1}^{m} a_i \geq 2^m - 1.$$

Proof:

- Now we show that in general $mx > \sum_{i=1}^{m} a_i \geq 2^m - 1$.
- The fact that $1 \leq a_i \leq x$ and $a_i < a_{i+1}$ implies that

$$\sum_{i=1}^{m} a_i < \sum_{i=1}^{m} x = mx.$$

- Let $A_1, \ldots, A_{2^m-1}$ denote the complete list of the $2^m - 1$ non-zero subsets of $\{a_1, \ldots, a_m\}$.
- Let

$$S_i = \sum_{a_j \in A_i} a_j$$

denote the sum of the elements in each A_i

Theorem 1

Want to show: $mx > \sum_{i=1}^{m} a_i \geq 2^m - 1$.

Proof:

- By assumption each of the S_i are distinct. So we may reorder the S_i so that

$$1 \leq S_1 < S_2 < \cdots < S_{2^m-1} < mx \tag{1}$$

- Note that we must have that $S_{2^m-1} = \sum_{i=1}^{m} a_i$.
- From equation 1 it follows that

$$S_i \geq i$$

and hence that

$$\sum_{i=1}^{m} a_i = S_{2^m-1} \geq 2^m - 1.$$

- Thus we have that

$$mx > \sum_{i=1}^{m} a_i \geq 2^m - 1.$$

□

Theorem 1

Theorem 1: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$mx > \sum_{i=1}^{m} a_i \geq 2^m - 1.$$

Corollary: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$2^m \leq mx$$

Corollary: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$\frac{2^m}{x} \leq m$$

Theorem 2

Theorem 2: If $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

(Sketch) Proof:

- We have equality if $a_i = 2^{i-1}$ for each i. (check using geometric sequence formula)
- Consider the sum of the squares of the 2^m quantities $\pm a_1 \pm a_2 \pm \cdots \pm a_m$
- Just to be clear, $a_1 - a_2 + a_3 + a_4 + \cdots + a_{m-2} - a_{m-1} - a_m$ and $-a_1 + a_2 + a_3 - a_4 + \cdots - a_{m-2} + a_{m-1} + a_m$ are just two examples of such quantities.
- We write the sum of the squares simply as $S = \sum(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2$.
- One can check that

$$S = \sum(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2 = 2^m(\sum_{i=1}^{m} a_i^2)$$

Theorem 2

Theorem 2: If $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

Proof:

- The 2^m quantities $\pm a_1 \pm a_2 \pm \cdots \pm a_m$.
  - They are distinct
  - Different from zero
  - Of the same parity (i.e. all either even or odd)
- By Theorem 1, each of the 2^m quantities lies between

$$-(2^m - 1) \leq \pm a_1 \pm a_2 \pm \cdots \pm a_m \leq 2^m - 1$$

- Hence

$$1 \leq (\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2 \leq (2^m - 1)^2$$

- The estimates above and the fact that the 2^m quantities are distinct, different from zero and of the same parity, implies the sum of their squares, S, is at least

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 \leq S$$

Theorem 2

Proof continued:

- We saw on the prev. slide that

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 \leq S$$

- Note now that

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 = 2\sum_{i=1}^{m}(2^i - 1)^2$$

- One can then check using basic results on the sums of geometric sequences that

$$2\sum_{i=1}^{m}(2^i - 1)^2 = \frac{1}{3}2^m(2^{2m} - 1).$$

- Thus we have that

$$\frac{1}{3}2^m(2^{2m} - 1) \leq S$$

Theorem 2

Proof continued:

- We saw on the prev. slide that

$$\frac{1}{3}2^m(2^{2m}-1) \leq S$$

- We also saw earlier that

$$S = 2^m(\sum_{i=1}^{m} a_i^2)$$

- Thus we've shown that

$$2^m \sum_{i=1}^{m} a_i^2 \geq \frac{2}{3}2^{m-1}(2^{2m}-1)$$

- Hence

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m-1)$$

as desired. □

A false conjecture

Recall -if $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

Theorem 1: $$\sum_{i=1}^{m} a_i \geq 2^m - 1.$$

Theorem 2: $$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

Conjecture:

$$\sum_{i=1}^{m} a_i^n \geq \frac{1}{2^n - 1}(2^{nm} - 1)$$

False: $n = 4$ yields a counterexample.

A false conjecture

Conjecture:

$$\sum_{i=1}^{m} a_i^4 \geq \frac{1}{15}(16^m - 1)$$

Falsity: The set of six numbers

$$\{a_i\} = \{11, 17, 20, 22, 23, 24\}$$

whose subsets have distinct sums. The sum of their fourth powers is 1 104 035, but $\frac{1}{15}(16^m - 1)$ for $m = 6$ is 1 118 481.

Theorem 3

Theorem 3: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$m < \log x + \frac{1}{2} \log \log x + 1.3$$

for $x \geq 2$. Here log means $\log_2$.

Proof:

- Note $1 \leq a_i \leq x$ implies $1 \leq a_i^2 \leq x^2$.
- Furthermore the fact that $a_i < a_{i+1}$ implies that

$$\sum_{i=1}^{m} a_i^2 < \sum_{i=1}^{m} x^2 = mx^2$$

Theorem 3

Theorem 3: If $a_1 < a_2 < \cdots < a_m \leq x$ are positive integers whose subsets have distinct sums then

$$m < \log x + \frac{1}{2}\log\log x + 1.3$$

for $x \geq 2$. Here log means $\log_2$.

Proof contd.:

- Theorem 2 then applies to show that

$$\frac{1}{3}(4^m - 1) \leq \sum_{i=1}^{m} a_i^2 < mx^2.$$

- Claim: This implies

$$4^m < 3mx^2$$

(see appendix for details)

Theorem 3

Proof contd.:

- Starting with $4^m < 3mx^2$ take log to the base 2 on either side.
- Then we see that
$$2m < \log 3mx^2 = \log 3m + 2\log x. \tag{2}$$
- Now $2^m \leq mx \implies m \leq \log(mx) \implies m \leq \log m + \log x$
- Also $m \leq x \implies \log m \leq \log x \implies m \leq \log x + \log x = 2\log x.$
- Using this we see that
$$\log 3m \leq \log(3 \cdot 2\log x) = \log(6\log x) = \log 6 + \log\log x$$
- Putting everything together we see that
$$2m < \log 6 + \log\log x + 2\log x.$$
- Now $\log 6 < 2.6$ and so we have that
$$2m < 2\log x + \log\log x + 2.6$$

and the result follows after diving by 2 on both sides. $\square$

First goal achieved

Q: What is the maximum number m of positive integers a_i satisfying $a_1 < a_2 < \cdots < a_m \leq x$ such that all possible sums of the a_i are distinct.

Bounds for m**:**

$$\lfloor \log_2 x \rfloor + 1 \leq m < \log x + \frac{1}{2} \log \log x + 1.3$$

where log here means log to the base 2.

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The Conway-Guy sequence

Definition (Conway-Guy Sequence): We define a sequence of integers $\{u_i\}_{i\in\mathbb{N}}$ in the following way:

- $u_0 = 0$
- $u_1 = 1$
- $u_{n+1} = 2u_n - u_{n-r}$ for $n \geq 1$,
  (where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$)

The Conway-Guy sequence

Definition (Conway-Guy Sequence): We define a sequence of integers $\{u_i\}_{i\in\mathbb{N}}$ in the following way: $u_0 = 0$; $u_1 = 1$ and

$$u_{n+1} = 2u_n - u_{n-r}$$

for $n \geq 1$, (where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$

Some values of u_n for small n:

n	u_n	u_{n-r}	$n-r$	r
1	1	0	0	1
2	2	0	0	2
3	4	1	1	2
4	7	1	1	3
5	13	2	2	3
6	24	4	3	3
7	44	4	3	4
8	84	7	4	4
9	161	13	5	4
10	309	24	6	4

The Conway-Guy sequence

Definition (Conway-Guy Sequence): We define a sequence of integers $\{u_i\}_{i\in\mathbb{N}}$ in the following way: $u_0 = 0$; $u_1 = 1$ and

$$u_{n+1} = 2u_n - u_{n-r}$$

for $n \geq 1$, (where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$

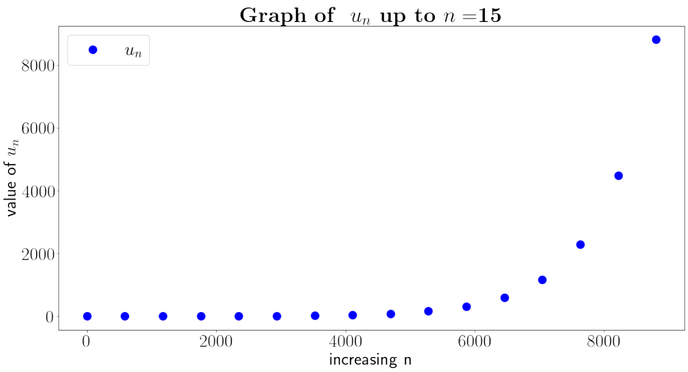
Some values of u_n for larger n:

n	u_n	u_{n-r}	$n-r$	r
22	1 051905	8807	15	7
23	2 095003	17305	16	7
24	4 172701	34301	17	7
25	8 311101	68008	18	7
26	16 554194	134852	19	7
27	32 973536	267420	20	7
28	65 679652	530356	21	7
29	130 828948	530356	21	8
30	261 127540	1 051905	22	8
31	521 203175	2 095003	23	8
32	1040 311347	4 172701	24	8
33	2076 449993	8 311101	25	8

The Conway-Guy sequence

Definition: We define a sequence of integers $\{u_i\}_{i\in\mathbb{N}}$ in the following way:

- $u_0 = 0$
- $u_1 = 1$
- $u_{n+1} = 2u_n - u_{n-r}$ for $n \geq 1$,
  (where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$)



The Conway-Guy sequence

Lemma 1: u_n is strictly increasing with n

Proof:

- The proof follows by induction
- As a base case we have that $u_1 = 1 > u_0 = 0$.
- Suppose that $u_{m+1} > u_m$ for all $0 \leq m \leq n$.
- We now show that $u_{n+1} > u_n$.
- By definition $u_{n+1} = 2u_n - u_{n-r}$
- We can rewrite this as $u_{n+1} - u_n = u_n - u_{n-r}$.
- Since $u_n > u_{n-r}$ by our induction hypothesis, we have that $u_{n+1} - u_n > 0$ which implies that $u_{n+1} > u_n$. $\square$

The Conway-Guy sequence

Lemma 2: $0 \leq u_n \leq 2^{n-1}$ for $n \geq 0$

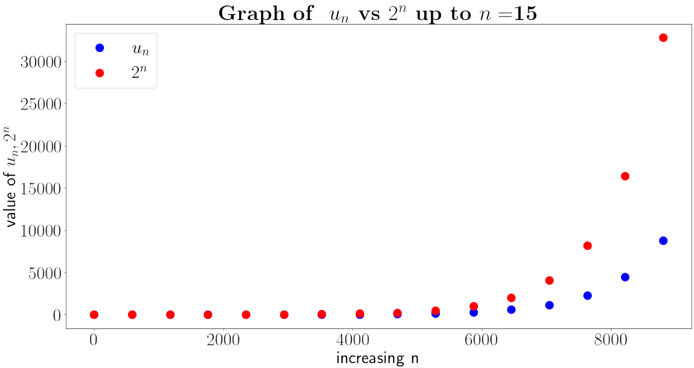
Proof:

- The proof follows by induction
- Base case: $0 = u_0 < 2^0 = 1$. Moreover $u_n \geq u_0 = 0$ since $\{u_i\}$ is strictly increasing by Lemma 1.
- Suppose $0 \leq u_m \leq 2^{m-1}$ for $0 \leq m \leq n$
- We will show $0 \leq u_{n+1} \leq 2^{n+1-1} = 2^n$.
- By definition $u_{n+1} = 2u_n - u_{n-r}$ for $n \geq 1$.
- Since $u_n > u_{n-r} > u_0 = 0$ (Lemma 1) we must have that $u_{n+1} \leq 2u_n$
- By the induction hypothesis $u_n \leq 2^{n-1}$, hence

$$u_{n+1} \leq 2u_n \leq 2 \cdot 2^{n-1} = 2^n$$

as desired. $\square$

The Conway-Guy sequence



The Conway-Guy sequence

Lemma 3: The sequence

$$\frac{u_n}{2^n}$$

is a decreasing function of n for $n \geq 1$ and strictly decreasing for $n \geq 4$.

Proof:

- For $n = 0$, $u_n = 0$, hence $\frac{u_0}{2^0} = 0$
- For $n = 1$, $u_n = 1$ and so $\frac{u_1}{2^1} = \frac{1}{2}$
- For $n = 2$, $u_n = 2u_1 - u_0 = 2$ and so $\frac{u_n}{2^n} = \frac{2}{2^2} = \frac{1}{2}$.
- For $n = 3$, $u_3 = 2u_2 - u_0 = 4$ and so $\frac{u_n}{2^n} = \frac{4}{2^3} = \frac{1}{2}$.

The Conway-Guy sequence

Lemma 3: The sequence

$$\frac{u_n}{2^n}$$

is a decreasing function of n for $n \geq 1$ and strictly decreasing for $n \geq 4$.

Proof:

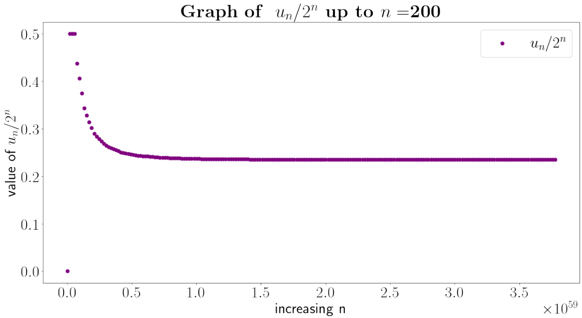
- Now by definition $u_{n+1} = 2u_n - u_{n-r}$ for $n \geq 1$.
- This implies that
$$\frac{u_{n+1}}{2^{n+1}} = \frac{u_n}{2^n} - \frac{u_{n-r}}{2^{n+1}}.$$
- For $n \geq 3$, we have that $r = \langle\sqrt{2n}\rangle < n$, so that $n - r > 0$ and $u_{n-r} > 0$.
- Thus for $n \geq 3$ we have that
$$\frac{u_{n+1}}{2^{n+1}} < \frac{u_n}{2^n}.$$
- Stated equivalently $\frac{u_n}{2^n}$ is strictly decreasing for $n \geq 4$. $\square$

The Conway-Guy sequence

Theorem 4: We have that

$$\lim_{n\to\infty} \frac{u_n}{2^n} = \alpha \quad \text{where} \quad 0 < \alpha < \frac{1}{2}.$$

Remark: In particular, this result implies that the sequence u_n behaves/grows like 2^n.
Proof: See Appendix.



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Main goal of the seminar

Recall the main goal of the seminar.

Q: What is the maximum number m of positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq x$$

such that all the 2^m sums of the a_i are distinct.

Main goal of the seminar: In the case for $x = 2^k$, we will show in this seminar that it is possible to have $m = k + 2$.

The sequence $\{a_i\}$

First recall the definition of the *Conway-Guy sequence*.

Definition (Conway-Guy Sequence): We define a sequence of integers $\{u_i\}_{i\in\mathbb{N}}$ in the following way:

- $u_0 = 0$
- $u_1 = 1$
- $u_{n+1} = 2u_n - u_{n-r}$ for $n \geq 1$,
  (where $r = \langle\sqrt{2n}\rangle$, the nearest integer to $\sqrt{2n}$)

The sequence $\{a_i\}$

Definition (Auxiliary Sequence): Using the Conway-Guy sequence, we define an auxiliary sequence $\{a_i\}$ of $k+2$ integers by setting

$$a_i = u_{k+2} - u_{k+2-i}$$

for $1 \leq i \leq k+2$.

The sequence $\{a_i\}$

Conjecture: Conway & Guy claim that the set of $k+2$ integers given by

$$A = \{a_i = u_{k+2} - u_{k+2-i} \mid 1 \leq i \leq k+2\}$$

has subsets with distinct sums. Conway and Guy also claim that A gives the best possible solution, that being $m = k+2$ to the problem.

Resolution: This conjecture was resolved. The above was proven to be true by Tom Bohman in 1996 in the paper - *"A Sum Packing Problem of Erdös and the Conway-Guy Sequence"* [a]

[a] See remarks below Theorem 1 in this paper, S_{n+1} there is the set A above.

The sequence $\{a_i\}$

Conjecture: Conway & Guy claim that the set of $k+2$ integers given by

$$A = \{a_i = u_{k+2} - u_{k+2-i} \mid 1 \le i \le k+2\}$$

has subsets with distinct sums and gives the best possible solution, that being $m = k+2$ to the problem.

Partial result: With the aid of the theorems that will soon be proved and increasing amounts of computational power it is possible to verify this conjecture for small values of k, for example for $k \le 40$.

We will see that $k \le 40$ is enough to achieve our main goal of the seminar.

The "trick" part 1.

Proposition (Trick): Given any set S of $k+2$ numbers each less than 2^k whose subsets have distinct sums, the set S' obtained by S by doubling each member and adding an odd number, i.e.

$$S' = \{2a \mid a \in S\} \cup \{m\} = 2S \cup \{m\}$$

where $m \in 2\mathbb{Z}+1$ has distinct sums.

The "trick" part 1.

Want to show: If S is a set with $|S| = k + 2$ and $\max S \leq 2^k$ then $S' = 2S \cup \{m\}$ where $m \in 2\mathbb{Z} + 1$ has distinct sums.

Proof:

- The subsets of $2S$ each yield distinct sums, since each sum is just 2 times the corresponding sum of elements in S and by assumption those sums are distinct. $(*)$
- Suppose now we have two subsets A and B of S' whose sum of their elements yield the same sum.
- If one subset of S' contains m and another subset does not, then their respective sums must be distinct since one sum is even and the other odd (a contradiction).
- If both subsets of S' contain m, we may simply remove m from the sum and fall into case $(*)$ again.
- Thus we've proven the claim. $\square$

The "trick" part 2.

Lemma (Trick): Given any set S of $k+2$ numbers each less than 2^k, whose subsets have distinct sums then for any positive integer l, the set

$$2^l S \cup \{2^i \mid 0 \leq i \leq l-1\}$$

has cardinality $k+2+l$ and also has distinct sums.

Proof:

- Let $S^{(1)} = 2S \cup \{1\}$. By the previous proposition this has subsets with distinct sums.
- Let $S^{(2)} = 2S^{(1)} \cup \{1\} = 2^2 S \cup \{2\} \cup \{1\}$. By the previous proposition this has subsets with distinct sums.
- Let $S^{(3)} = 2S^{(2)} \cup \{1\} = 2^3 S \cup \{2^2, 2\} \cup \{1\}$. By the previous proposition this has subsets with distinct sums.
- Continue inductively to obtain

$$S^{(l)} = 2S^{(l-1)} \cup \{1\} = 2^l S \cup \{2^i \mid 0 \leq i \leq l-1\}$$

which also has distinct sums by the previous proposition. $\square$

Main goal of the seminar

Q: What is the maximum number m of positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq x$$

such that all the 2^m sums of the a_i are distinct.

Main goal of the seminar: In the case for $x = 2^k$, we will show in this seminar that it is possible to have $m = k + 2$.

Claim: The sequence $\{a_i\}$ we've defined along with the two tricks will give the above result.

Main goal of the seminar

How it goes:

- Consider the sequence $\{a_i = u_{k+2} - u_{k+2-i}\}$.
- Recall earlier we defined $\alpha_n := \frac{u_n}{2^n}$. One can verify by hand/computation that

$$\alpha_{23} = \frac{u_{23}}{2^{23}} < \frac{1}{4} = 2^{-2}$$

  Moreover we know that α_n is a strictly decreasing sequence for $n \geq 4$ by Lemma 3. Hence $\alpha_k = \frac{u_k}{2^k} < 2^{-2}$ for $k \geq 23$

- Then $\frac{u_k}{2^k} < 2^{-2}$ for $k \geq 23$ implies that we have $u_{k+2} \leq 2^k$ for $k \geq 21$.
- Thus $a_i \leq 2^k$ for $k \geq 21$.

Main goal of the seminar

How it goes: (continued)

- Let $x = 2^k$ be given for $k \geq 21$.
- Pick $z = 21$ (for simplicity)
- Consider the set $A = \{a_i = u_{k+2} - u_{k+2-i} \mid 1 \leq i \leq z+2\}$
- One can verify by *computation* that A has subsets with distinct sums.
- We have $a_i \leq 2^z$ for each $a_i \in A$.
- Let $l = k - z$.
- Then the set
$$A' := 2^l A \cup \{2^i \mid 0 \leq i \leq l-1\}$$
has cardinality $(z+l)+2 = k+2$ and also has distinct sums by the previous Lemma (trick).
- Moreover if $a \in A'$ then $a \leq 2^k$

Anecdote: I managed to verify that A had distinct sums for $z = 23$ before my 16GB of RAM could not take any more.

Main goal of the seminar - completed

Q: What is the maximum number m of positive integers a_i satisfying

$$a_1 < a_2 < \cdots < a_m \leq x$$

such that all the 2^m sums of the a_i are distinct.

Main goal of the seminar: In the case for $x = 2^k$, we will show in this seminar that it is possible to have $m = k + 2$.

We just showed on the prev. slide that we can have $m = k + 2$

A shift in the seminar

- We now turn to proving results that are useful towards the conjecture made by Conway & Guy.

Conjecture: Conway & Guy claim that the set of $k+2$ integers given by

$$A = \{a_i = u_{k+2} - u_{k+2-i} \mid 1 \leq i \leq k+2\}$$

has subsets with distinct sums and also claim that A gives the best possible solution, that being $m = k+2$ to the problem.

- Alternatively you can view everything that follows as us basically proving a lot of properties of the *Conway-Guy* sequence.

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Lemma 4

Lemma 4: For $n \geq 1$ we have that $u_{n+1} > u_n + u_{n-1}$.

Proof:

- For $n = 1$, we have that $u_2 = 2 > u_1 + u_0 = 1 + 0 = 1$.
- For $n = 2$, we have that $u_3 = 4 > u_2 + u_1 = 2 + 1 = 3$.
- For $n = 3$, we have that $u_4 = 7 > u_3 + u_2 = 4 + 2 = 6$.
- For $n = 4$, we have that $u_5 = 13 > u_4 + u_3 = 7 + 4 = 11$.

Some values of u_n for small n:

n	u_n	u_{n-r}	$n-r$	r
1	1	0	0	1
2	2	0	0	2
3	4	1	1	2
4	7	1	1	3
5	13	2	2	3
6	24	4	3	3
7	44	4	3	4
8	84	7	4	4
9	161	13	5	4
10	309	24	6	4

Lemma 4

Lemma 4: For $n \geq 1$ we have that $u_{n+1} > u_n + u_{n-1}$.

Proof continued:

- Induction hypothesis: Suppose that for $n-1 \geq m \geq 1$ we have that $u_{m+1} > u_m + u_{m-1}$.
- Now suppose $n \geq 4$, then in particular $\sqrt{2n} \geq \sqrt{8} = 2 \cdot \sqrt{2} > 2$. This implies that $r = \langle \sqrt{2n} \rangle > 2$ and in particular that $n - r < n - 2$ and that $u_{n-r} < u_{n-2}$.
- Then by definition we know that $u_{n+1} = 2u_n - u_{n-r} = u_n + (u_n - u_{n-r}) > u_n + (u_n - u_{n-2})$
- From the induction hypothesis we know that $u_n > u_{n-1} + u_{n-2}$ which implies that $u_n - u_{n-2} > u_{n-1}$
- This implies that
$$u_{n+1} > u_n + u_{n-1}$$
as desired. $\square$

Lemma 5

Lemma 5: For $n \geq 4$ we have that

$$u_{n+1} < \sum_{i=0}^{n} u_i \leq u_{n+1} + u_{n-2}.$$

Proof: See Appendix

Equal sums warm-up

Proposition: There are no singletons, pairs, triples or quadruples of the u_i with equal sums

- **Singletons:** Just note that $\{u_i\}$ is a strictly increasing sequence.
- **Pairs:** Follows from the fact that

$$u_{n+1} > u_n + u_{n-1}$$

- **Triples:** Follows from the fact that

$$u_{n+1} \geq u_n + u_{n-1} + u_{n-2} \text{ for } n \geq 2$$

- **Quadruples:** Follows from the fact that

$$u_{n+1} \geq u_n + u_{n-1} + u_{n-2} + u_{n-3} \text{ for } n \geq 11$$

Proofs: See Appendix.

Theorem 5

Theorem 5: If two subsets of the $\{a_i\}$ have equal sums, then there are two subsets of the $\{u_i\}$ with equal sums and equal cardinalities. Conversely if there are two subsets of the $\{u_i\}$ with equal sums then there are two subsets of the $\{a_i\}$ with equal sums and equal cardinalities.

- For cardinalities less than 4 the theorem is vacuously true by the preceding Lemmas.
- We now prove it for $k + 1 \geq 4$.

Theorem 5

Theorem 5: If two subsets of the $\{a_i\}$ have equal sums, then there are two subsets of the $\{u_i\}$ with equal sums and equal cardinalities. Conversely if there are two subsets of the $\{u_i\}$ with equal sums then there are two subsets of the $\{a_i\}$ with equal sums and equal cardinalities.

Proof:

- Suppose that two subsets of the $\{a_i\}$ have equal sums. Denote these sets by $\{a_{i_1}, \ldots, a_{i_s}\}$ and $\{a_{j_1}, \ldots, a_{j_t}\}$.
- Since $a_i = u_{k+2} - u_{k+2-i}$ we have that

$$(u_{k+2} - u_{k+2-i_1}) + \cdots + (u_{k+2} - u_{k+2-i_s}) = (u_{k+2} - u_{k+2-j_1}) + \cdots + (u_{k+2} - u_{k+2-j_t}) \tag{3}$$

- We may assume that (i) the two sets are disjoint (else we could just cancel common terms); (ii) that $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$ and (iii) that $s \geq t$.
- By rearranging equation (3) we arrive at

$$(s-t)u_{k+2} = u_{i_1} + u_{i_2} + \cdots + u_{i_s} - (u_{j_1} + \cdots u_{j_t}) \tag{4}$$

Theorem 5

Theorem 5: If two subsets of the $\{a_i\}$ have equal sums, then there are two subsets of the $\{u_i\}$ with equal sums and equal cardinalities. Conversely if there are two subsets of the $\{u_i\}$ with equal sums then there are two subsets of the $\{a_i\}$ with equal sums and equal cardinalities.

Proof:

- On the prev. slide we arrived at equation 4 which says that

$$(s-t)u_{k+2} = u_{i_1} + u_{i_2} + \cdots + u_{i_s} - (u_{j_1} + \cdots u_{j_t})$$

- Since $1 \le i_m \le k+1$ for $1 \le m \le s$, we have that $u_0 \le u_{i_m} \le u_{k+1}$ and hence that $u_{i_1} + u_{i_2} + \cdots + u_{i_s} \le \sum_{i=0}^{k+1} u_i$ which implies that the RHS of equation 4 above is *strictly!* less than $\sum_{i=0}^{k+1} u_i$.
- Now by Lemma 5, we know that $\sum_{i=0}^{k+1} u_i \le u_{k+2} - u_{k+1}$, hence we see that

$$(s-t)u_{k+2} < u_{k+2} - u_{k+1}.$$

- Thus

$$s - t < \frac{u_{k+2} - u_{k+1}}{u_{k+2}} = 1 - \frac{u_{k+1}}{u_{k+2}} < 2.$$

- Thus we either have $s-t=0$ or $s-t=1$, i.e. $s=t$ or $s=t+1$

Theorem 5

Proof continued:

- Recall equation 4 which says that

$$(s-t)u_{k+2} = u_{i_1} + u_{i_2} + \cdots + u_{i_s} - (u_{j_1} + \cdots u_{j_t})$$

- If $s = t$ then from equation 4 above we simply have

$$u_{i_1} + u_{i_2} + \cdots + u_{i_s} = u_{j_1} + \cdots u_{j_t}$$

  and so we obtain subsets of the $\{u_i\}$ with equal sums and equal cardinalities.

- If $s = t + 1$ then by rearranging equation 4 above we have that

$$u_{i_1} + u_{i_2} + \cdots + u_{i_s} = u_{j_1} + \cdots u_{j_t} + u_{k+2}$$

  and again we obtain subsets of the $\{u_i\}$ with equal sums and equal cardinalities, this time the cardinality of both sets is $s + 1$.

Theorem 5

Proof continued:

- Now conversely suppose that there are two subsets $\{u_{i_1}, \ldots, u_{i_s}\}$ and $\{u_{j_1}, \ldots, u_{j_s}\}$ of the $\{u_i\}$ with equal sums and cardinalities.
- We can assume without loss of generality that $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$.
- Then we have
$$u_{i_1} + \cdots + u_{i_s} = u_{j_1} + \cdots + u_{j_s}$$
- We can rewrite each i_m, j_m as $i_m = k + 2 - i'_m$ and $j_m = k + 2 - j'_m$ for $1 \leq m \leq s$.
- Thus
$$u_{k+2-i'_1} + \cdots + u_{k+2-i'_s} = u_{k+2-j'_1} + \cdots + u_{k+2-j'_s}$$

Theorem 5

Proof continued:

- We saw on the prev. slide that

$$u_{k+2-i'_1} + \cdots + u_{k+2-i'_s} = u_{k+2-j'_1} + \cdots + u_{k+2-j'_s}$$

- Then for any $n > \max(k+2-i'_s, k+2-j'_s)$ we have that

$$(u_n - u_{k+2-i'_1}) + \cdots + (u_n - u_{k+2-i'_s}) = (u_n - u_{k+2-j'_1}) + \cdots + (u_n - u_{k+2-j'_s})$$

- In particular if $n = k+2$ we then obtain that

$$(u_{k+2} - u_{k+2-i'_1}) + \cdots + (u_{k+2} - u_{k+2-i'_s}) = (u_{k+2} - u_{k+2-j'_1}) + \cdots + (u_{k+2} - u_{k+2-j'_s})$$

- This is the same as saying that

$$a_{i'_1} + \cdots + a_{i'_s} = a_{j'_1} + \cdots + a_{j'_s}$$

which completes the proof. $\square$

Triangular numbers and the Conway-Guy sequence

Definition: Triangular numbers are given in the form

$$T_s = \frac{1}{2}s(s+1)$$

- If $r = \langle\sqrt{2n}\rangle$, then
$$T_{r-1} < n \leq T_r$$
for $n > 0$
- For u_{T_s} we have that $r = \langle\sqrt{s(s+1)}\rangle \sim s$. Thus
$$u_{T_s} \sim 2u_{T_s-1} - u_{T_s-s}$$
- We have the identity
$$u_{T_{s+1}+t+1} = 2u_{T_{s+1}+t} - u_{T_s+t-1}$$
where $1 \leq t \leq s+2$

Theorem 6

Theorem 6: If $T_s = \frac{1}{2}s(s+1)$, $s \geq 0$ and $0 \leq t \leq s+2$, then

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i}.$$

(If $s = 1$ or $s = 0$, interpret the empty or 'less than empty' sum on the right as 0 or -1 respectively.)

Example: If $s = 3$, then $T_s = 6$ and $T_{s+1} = T_4 = \frac{1}{2}(4)(5) = 10$ and the theorem says that for $0 \leq t \leq 5$ we have that

$$\sum_{i=t+6}^{t+10} u_i = u_{t+11} + (u_3 + u_6).$$

The left hand side is the set $\{u_{t+6}, u_{t+7}, u_{t+8}, u_{t+9}, u_{t+10}\}$ of cardinality $s+2 = 5$ and the right hand side is the set $\{u_{t+11}, u_3, u_6\}$ of cardinality $s = 3$.

Importance of Theorem 6

Theorem 6: If $T_s = \frac{1}{2}s(s+1)$, $s \geq 0$ and $0 \leq t \leq s+2$, then

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i}.$$

(If $s = 1$ or $s = 0$, interpret the empty or 'less than empty' sum on the right as 0 or -1 respectively.)

Remark: Theorem 6 exhibits sets of the u_i with equal sums whose cardinalities are $s+2$ (on the LHS) and either s or $s+1$ (on the RHS).

Theorem 6

Theorem 6: If $T_s = \frac{1}{2}s(s+1)$, $s \geq 0$ and $0 \leq t \leq s+2$, then

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i}.$$

(If $s=1$ or $s=0$, interpret the empty or 'less than empty' sum on the right as 0 or -1 respectively.)

Proof:

- The theorem may be verified by hand from Table 1 for $s = 0, 1, 2$ and $0 \leq t \leq s+2$.
- Note that the result for $t = s+2$ is the same as that for $t = 0$ and $s+1$ in place of s, if we add u_{T_s+1} to each side. This is because $T_{s+1} + s + 2 = T_{s+2}$ and $T_s + s + 2 = T_{s+1} + 1$ (one can verify this by routine algebra) imply that

$$\sum_{i=T_s+s+2}^{T_{s+1}+s+2} u_i = \sum_{i=T_{s+1}+1}^{T_{s+2}} u_i$$

Theorem 6

Proof:

- *Induction hypothesis:* We assume the result holds true for some $s \geq 2$ and some t with $0 \leq t \leq s+1$ and we prove that it is true for the same s and for $t+1$ in place of t.
- So by assumption we have that

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i}.$$

- Then we add $u_{T_{s+1}+t+1} - u_{T_s+t}$ to either side to get

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i + u_{T_{s+1}+t+1} - u_{T_s+t} = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} + u_{T_{s+1}+t+1} - u_{T_s+t}.$$

- This implies that

$$\sum_{i=T_s+t+1}^{T_{s+1}+t+1} u_i = 2u_{T_{s+1}+t+1} - u_{T_s+t} + \sum_{i=2}^{s} u_{T_i}.$$

Theorem 6

Proof:

- We have from prev. slide
$$\sum_{i=T_s+t+1}^{T_{s+1}+t+1} u_i = 2u_{T_{s+1}+t+1} - u_{T_s+t} + \sum_{i=2}^{s} u_{T_i}.$$
- By definition of the sequence of the u_i we have that $2u_{T_{s+1}+t+1} - u_{T_s+t} = u_{T_{s+1}+t+2}$ for $0 \leq t \leq s+1$.
- This implies that
$$\sum_{i=T_s+t+1}^{T_{s+1}+t+1} u_i = u_{T_{s+1}+t+2} + \sum_{i=2}^{s} u_{T_i}$$
which is what we wanted. $\square$

Lemmas 8 & 9

Lemma 8: If $s \geq 0$, with the convention of Theorem 6,

$$\sum_{i=2}^{s} u_{T_i} < \frac{1}{2}\left(u_{T_s+1} + u_{T_{s-1}+2}\right)$$

Lemma 9: If $v > T_{s+1}$, then

$$\sum_{i=v-s}^{v} u_i < u_{v+1}.$$

They generalize...

Lemma 5: For $n \geq 4$ we have that

$$u_{n+1} < \sum_{i=0}^{n} u_i \leq u_{n+1} + u_{n-2}.$$

Theorem 7

Theorem 7: If $s \geq 0$ and $1 \leq t \leq s+2$, then with the same convention as in Theorem 6,

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Proof: See appendix. □

It generalizes...

Lemma 4: For $n \geq 1$ we have that $u_{n+1} > u_n + u_{n-1}$.

Theorem 7

Theorem 7: If $s \geq 0$ and $1 \leq t \leq s+2$, then with the same convention as in Theorem 6,

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Example: For $s = 4$, we have that $T_4 = \frac{1}{2}(4)(5) = 10$ and $1 \leq t \leq 6$. Then the theorem says that

$$u_{t+11} > \sum_{i=0}^{t+9} u_i + u_3 + u_6 + u_{10}.$$

Theorem 8

Theorem 8: Suppose there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$.

Then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Remark: This theorem is not vacuous since there *are* sets of the u_i with equal sums, but which do not have the same cardinality (cf. Theorem 6).

Theorem 8

Theorem 8: Suppose there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$.

Then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Example: Take $s = 4$ in the above, then we have $T_s = 10$ that $T_{s+1} = 15$. The largest member of either set is $u_{T_{s+1}+t+1} = u_{t+16}$ where $1 \leq t \leq 6$. The other contains the $s+1 = 5$ members u_i for i in the range $t + 11 \leq i \leq t + 15$. Taking $t = 3$ implies that one set contains as its largest member u_{19} and the other contains $\{u_{14}, u_{15}, u_{16}, u_{17}, u_{18}\}$.

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof:

- Call the sets of the u_i which have equal sums A and B.
- Let S_1 be the sum of the elements of A and S_2 be the sum of the elements of B. We have $S_1 = S_2$ by assumption.
- Suppose B does not contain the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.
- Then the sum of the elements of B is at most

$$S_2 \leq \sum_{i=0}^{T_{s+1}+t} u_i - \sum_{T_s+t+1}^{T_{s+1}+t} u_i.$$

- Now certainly we have that

$$\sum_{i=0}^{T_{s+1}+t} u_i - \sum_{T_s+t+1}^{T_{s+1}+t} u_i < \sum_{i=0}^{T_{s+1}+t} u_i - u_{T_s+t+1}$$

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s+t+1 \leq i \leq T_{s+1}+t$.

Proof contd.:

- We saw on the prev. slide that

$$S_2 < \sum_{i=0}^{T_{s+1}+t} u_i - u_{T_s+t+1}$$

- This since $\sum_{i=0}^{T_{s+1}+t} u_i = \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=T_s+t}^{T_{s+1}+t} u_i$ the above implies that

$$S_2 < \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=T_s+t}^{T_{s+1}+t} u_i - u_{T_s+t+1}$$

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof contd.:

- Continuing from prev. slide:

$$\begin{aligned} S_2 &< \sum_{i=T_s+t}^{T_{s+1}+t} u_i - u_{T_s+t+1} + \sum_{i=0}^{T_s+t-1} u_i \\ &= u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t+1} + \sum_{i=0}^{T_s+t-1} u_i \text{ (by Thm. 6)} \\ &< u_{T_{s+1}+t+1} + u_{T_s+t+1} - u_{T_s+t+1} \text{ (by Thm. 7)} \\ &= u_{T_{s+1}+t+1} \end{aligned}$$

- Now $u_{T_{s+1}+t+1}$ is either:
  - an element of A in which case $S_1 \geq u_{T_{s+1}+t+1}$ and we have that $u_{T_{s+1}+t+1} \leq S_1 = S_2 < u_{T_{s+1}+t+1}$ a contradiction
  - an element of B in which case $u_{T_{s+1}+t+1}$ is a summand of S_2 and so we also have a contradiction.

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof contd.:

- So we've shown that B must contain the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof contd.:

- Now we show that B must contain $s+2$ members.
- Now we can write the sum R of the $s+1$ members as

$$R = \sum_{i=T_s+t+1}^{T_{s+1}+t} u_i = \sum_{i=T_s+t}^{T_{s+1}+t} u_i - u_{T_s+t}$$

- From Theorem 6 we know that

$$\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i}$$

- Using the equality derived from Theorem 6 above we see that

$$R = u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t}$$

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof contd.:

- Then we can apply Lemma 8 to see that

$$\sum_{i=2}^{s} u_{T_i} < \frac{1}{2}(u_{T_s+1} + u_{T_{s-1}+2})$$

- Thus

$$\begin{aligned} R &= u_{T_{s+1}+t+1} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+t} \\ &< u_{T_{s+1}+t+1} + \frac{1}{2}(u_{T_s+1} + u_{T_{s-1}+2}) - u_{T_s+t} \\ &= u_{T_{s+1}+t+1} - \frac{1}{2}(2u_{T_s+t} - u_{T_s+1} - u_{T_{s-1}+2}) \\ &< u_{T_{s+1}+t+1} \end{aligned}$$

Theorem 8

Theorem 8: If there are two sets of the u_i with equal sums and the largest member of either set is $u_{T_{s+1}+t+1}$ where $1 \leq t \leq s+2$, then the other set contains at least $s+2$ members, including the $s+1$ members u_i for i in the range $T_s + t + 1 \leq i \leq T_{s+1} + t$.

Proof contd.:

- We saw on the previous slide that
$$R < u_{T_{s+1}+t+1}.$$
- Now remember S_2 is the sum of all the elements in B and S_1 is the sum of all the elements in A and we have that
$$S_1 = S_2$$
.
- Either A or B contains $u_{T_{s+1}+t+1}$ and the fact that $R < u_{T_{s+1}+t+1}$ (where R is the sum over $s+1$ elements of B) implies that B must contain at least one other element in addition to the $s+1$ elements which comprise the sum R in order for $S_1 = S_2$ to hold.
- Thus B has at least $s+2$ elements. □

Conditions for Theorems 9-13

- In Theorems 9-13 we will assume some extra conditions.
- Two of these conditions will be that there are two sets of the u_i with equal sums and equal cardinalities.
- Theorem 5 then would imply that the sequence $\{a_i\}$ will have equal sums. This is opposite to the conjecture made by Conway and Guy.
- So the author of the paper conjectures that these theorems **are only vacuously true**.

Conditions for Theorems 9-13

Conditions C and D are of 'minimal criminal' type as the author puts it.

Condition A: There are two sets of the u_i with equal sums

Condition B: The two sets have the same cardinality c

Condition C: Of such pairs of sets we choose one with the least possible greatest element u_{n+1} and write n in the form $T_{s+1} + t$ where $1 \leq t \leq s + 2$.

Condition D: Among pairs of sets satisfying conditions A to C, choose one with the smallest value of c. This condition implies that the two sets are disjoint. Lemmas 1, 4, 6 and 7 imply that $c \geq 5$.

Minor and Major sets

Definition: Suppose we have two sets of the u_i with equal sums. We call the set containing u_{n+1} the *major set* and the other set the *minor set*.

Theorem 9: Under conditions A to D, u_{T_s+t-1} belongs to the minor set.

Theorem 10: Under conditions A to D, the minor set does not contain all the $s+4$ members u_i for $T_s+t-2 \leq i \leq T_{s+1}+t$.

Proofs: See Appendix

Theorems 12 & 13

Theorem 11: Under conditions A to D, u_{T_s+t} belongs to neither set.

Theorem 12: If $s \geq 4$ and $1 \leq t \leq s$, the minor set contains u_i for $T_s + 1 \leq i \leq T_s + t - 1$. If $t = s + 1$ or $t = s + 2$, the minor set contains u_i for $T_s + 2 \leq i \leq T_s + t - 1$.

Theorem 13: The minor set contains u_x where $x = T_{s-1}$ if $t = 1$ (or 2) and $x = T_{s-1} + t - 2$ if $2 \leq t \leq s + 2$

Table of Contents

Geometric sequence

Formula for geometric sequence used: If we have a geometric sequence $\{r^0, r^1, \ldots, r^{n-1}\}$ that

$$\sum_{i=1}^{n} cr^{k-1} = \frac{c(1-r^n)}{1-r}$$

Theorem 2

Theorem 2: If $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

Proof:

- Consider the sum of the squares of the 2^m quantities $\pm a_1 \pm a_2 \pm \cdots \pm a_m$
- Just to be clear, $a_1 - a_2 + a_3 + a_4 + \cdots + a_{m-2} - a_{m-1} - a_m$ and $-a_1 + a_2 + a_3 - a_4 + \cdots - a_{m-2} + a_{m-1} + a_m$ are just two examples of such quantities.
- We write the sum of the squares simply as $S = \sum(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2$.
- Let's try and find a simpler expression for $S = \sum(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2$.
- Now consider (for the moment) $m = 2$. We have $2^2 = 4$ quantities
    - $a_1 + a_2$
    - $-a_1 + a_2$
    - $a_1 - a_2$
    - $-a_1 - a_2$
- What is $\sum(\pm a_1 \pm a_2)^2$? Let's investigate

Theorem 2

Proof contd.:

- We have the $2^2 = 4$ quantities

$$a_1 + a_2 \; ; \; -a_1 + a_2 \; ; \; a_1 - a_2 \; ; \; -a_1 - a_2$$

  and we want to know what is $\sum(\pm a_1 \pm a_2)^2$?
- We have the following values for $(\pm a_1 \pm a_2)^2$
  - $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$
  - $(-a_1 + a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2$
  - $(a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2$
  - $(-a_1 - a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$
- Then we see that

$$\begin{aligned}\sum(\pm a_1 \pm a_2)^2 &= (a_1 + a_2)^2 + (-a_1 + a_2)^2 + (a_1 - a_2)^2 + (-a_1 - a_2)^2 \\ &= 4(a_1^2 + a_2^2) \\ &= 2^2(\sum_{i=1}^{2} a_i^2)\end{aligned}$$

Theorem 2

Theorem 2: If $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

Proof contd.:

- For $m = 2$ we had that

$$\sum(\pm a_1 \pm a_2)^2 = 2^2(\sum_{i=1}^{2} a_i^2)$$

  this generalizes and we have that

$$S = \sum(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2 = 2^m(\sum_{i=1}^{m} a_i^2)$$

Theorem 2

Proof:

- Recall the 2^m quantities $\pm a_1 \pm a_2 \pm \cdots \pm a_m$.
- Claim: They are all distinct.
- Suppose to the contrary that two of them are equal, then that means that

$$\sum_{\substack{i \in I \\ |I| < m}} a_i - \sum_{\substack{i \in J \\ |J| < m}} a_i = \sum_{\substack{i \in K \\ |K| < m}} a_i - \sum_{\substack{i \in L \\ |L| < m}} a_i$$

  where $|I| + |J| = m$ and $|K| + |L| = m$.
- If $I \cap K \neq \emptyset$ or $J \cap L = \emptyset$, then they have a common term and we can cancel it from the sum and then use induction to prove that $I = K$ and $J = L$ and the result follows.
- Otherwise $I \cap K = \emptyset$ and $J \cap L = \emptyset$ and in this case we can rearrange to get

$$\sum_{\substack{i \in I \\ |I| < m}} a_i + \sum_{\substack{i \in L \\ |L| < m}} a_i = \sum_{\substack{i \in K \\ |K| < m}} a_i + \sum_{\substack{i \in J \\ |J| < m}} a_i$$

  and the LHS cannot equal the RHS because the a_i have distinct sums by assumption so we get a contradiction.

Theorem 2

Theorem 2: If $a_1 < a_2 < \cdots < a_m$ are positive integers whose subsets have distinct sums then

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m - 1).$$

Proof:

- Recall the 2^m quantities $\pm a_1 \pm a_2 \pm \cdots \pm a_m$.
  - They are distinct
  - Different from zero
  - Of the same parity (i.e. all either even or odd)
- By Theorem 1, each of the 2^m quantities lies between

$$-(2^m - 1) \leq \pm a_1 \pm a_2 \pm \cdots \pm a_m \leq 2^m - 1$$

- Hence

$$(\pm a_1 \pm a_2 \pm \cdots \pm a_m)^2 \leq (2^m - 1)^2$$

- The estimates above and the fact that the 2^m quantities are distinct, different from zero and of the same parity, implies the sum of their squares, S, is at least

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 \leq S$$

Theorem 2

Proof continued:

- We saw on the prev. slide that

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 \leq S$$

- Note now that

$$1^2 + (-1)^2 + 3^3 + (-3)^2 + \cdots + (2^m - 1)^2 + (1 - 2^m)^2 = 2\sum_{i=1}^{m}(2^i - 1)^2$$

- One can then check using basic results on the sums of geometric sequences that

$$2\sum_{i=1}^{m}(2^i - 1)^2 = \frac{2}{3}2^{m-1}(2^{2m} - 1).$$

- Thus we have that

$$\frac{2}{3}2^{m-1}(2^{2m} - 1) \leq S$$

Theorem 2

Proof continued:

- We saw on the prev. slide that

$$\frac{2}{3}2^{m-1}(2^{2m}-1) \leq S$$

- We also saw earlier that

$$S = 2^m(\sum_{i=1}^{m} a_i^2)$$

- Thus we've shown that

$$2^m \sum_{i=1}^{m} a_i^2 \geq \frac{2}{3}2^{m-1}(2^{2m}-1)$$

- Hence

$$\sum_{i=1}^{m} a_i^2 \geq \frac{1}{3}(4^m-1)$$

as desired. □

Theorem 3

Proof contd.:

- Theorem 2 then applies to show that

$$\frac{1}{3}(4^m - 1) \leq \sum_{i=1}^{m} a_i^2 < mx^2.$$

- Claim: $\frac{1}{3}(4^m - 1) \leq \sum_{i=1}^{m} a_i^2$ implies that $4^m < 3mx^2$
- In order to prove this we have two cases to examine.
- Case 1: We need to check that if $\frac{1}{3}(4^m - 1) = \sum_{i=1}^{m} a_i^2$ then $4^m < 3mx^2$
- Case 2: We need to check that if $\frac{1}{3}(4^m - 1) < \sum_{i=1}^{m} a_i^2$ then $4^m < 3mx^2$

Theorem 3

Want to show: <u>Case 1:</u> If $\frac{1}{3}(4^m - 1) = \sum_{i=1}^m a_i^2$ then $4^m < 3mx^2$

Proof contd.:

- If $\frac{1}{3}(4^m - 1) = \sum_{i=1}^m a_i^2$ then by Theorem 2 we have that $a_i = 2^{i-1}$ for each i. Moreover in this case we have that $x = 2^{m-1}$.
- Thus we have

$$\begin{aligned}
3mx^2 &> 2mx^2 \\
&= 2m(2^{m-1})^2 \\
&= m2^{2m-1} \\
&\geq \left(\frac{2^m}{2^{m-1}}\right) 2^{2m-1} \quad \text{since } m \geq \frac{2^m}{x} \text{ and } x = 2^{m-1} \\
&= 2 \cdot 2^{2m-1} \\
&= 2^{2m} \\
&= 4^m
\end{aligned}$$

Theorem 3

Want to show: Case 2: If $\frac{1}{3}(4^m - 1) < \sum_{i=1}^m a_i^2$ then $4^m < 3mx^2$

Proof contd.:

- If
$$\frac{1}{3}(4^m - 1) < \sum_{i=1}^m a_i^2 < mx^2,$$
since we are only working with integers we then see that
$$\frac{1}{3}(4^m - 1) \leq \sum_{i=1}^m a_i^2 - 1 \leq mx^2 - 1.$$
- Forget about the center term in this inequality and multiply by 3 throughout to see that
$$4^m - 1 \leq 3mx^2 - 3.$$
- From this we get that $4^m < 3mx^2$.
- Thus the claim is proven and we have in all cases that $4^m < 3mx^2$.

Lemma 5

Lemma 5: For $n \geq 4$ we have that

$$u_{n+1} < \sum_{i=0}^{n} u_i \leq u_{n+1} + u_{n-2}.$$

Proof:

- We first show this explicitly for $n = 4, 5$ and 6.
- $n = 4$:
  - $u_5 = 13$
  - $u_0 + u_1 + u_2 + u_3 + u_4 = 0 + 1 + 2 + 4 + 7 = 14$
  - $u_5 + u_2 = 15$
  - So $u_5 < \sum_{i=0}^{4} u_i \leq u_5 + u_2$.
- $n = 5$ and $n = 6$ can check explicitly similarly.

Lemma 5

Proof continued:

- Suppose that $u_{n+1} < \sum_{i=0}^{n} u_i \leq u_{n+1} + u_{n-2}$ is true for $n = k \geq 6$, we will show that it is true for $k+1$.
- We see that
$$\sum_{i=0}^{k+1} u_i = u_{k+1} + \sum_{i=0}^{k} u_i > u_{k+1} + u_{k+1} = 2u_{k+1}$$
with the last inequality occurring because $\sum_{i=0}^{k} u_i > u_{k+1}$ by the induction hypothesis.
- Now because $n = k \geq 6$ we see that $u_{(k+1)-r} > 0$ (where $r = \langle \sqrt{2(k+1)} \rangle$) so that $2u_{k+1} > 2u_{k+1} - u_{(k+1)-r} = u_{k+2}$.
- Hence
$$\sum_{i=0}^{k+1} u_i > u_{k+2}$$

Lemma 5

Proof continued:

- We saw on the last slide that

$$\sum_{i=0}^{k+1} u_i > u_{k+2}$$

- Now also using the induction hypothesis on $\sum_{i=0}^{k} u_i$ we see that

$$\sum_{i=0}^{k+1} u_i = u_{k+1} + \sum_{i=0}^{k} u_i \leq u_{k+1} + u_{k+1} + u_{k-2}.$$

- Now

$$\begin{aligned} u_{k+1} + u_{k+1} + u_{k-2} &= 2u_{k+1} + u_{k-2} \\ &= \left(2u_{k+1} - u_{(k+1)-r}\right) + u_{(k+1)-r} + u_{k-2} \\ &= u_{k+2} + u_{(k+1)-r} + u_{k-2}. \end{aligned}$$

- Hence

$$\sum_{i=0}^{k+1} u_i \leq u_{k+2} + u_{(k+1)-r} + u_{k-2}.$$

Lemma 5

Proof continued:

- We saw that
$$\sum_{i=0}^{k+1} u_i \leq u_{k+2} + u_{(k+1)-r} + u_{k-2}.$$
- Now for $k \geq 6$ we have that $(k+1) - r < k - 2$, so by Lemma 4
$$u_{(k+1)-r} + u_{k-2} < u_{k-1}$$
- Thus
$$u_{k+2} + u_{(k+1)-r} + u_{k-2} < u_{k+2} + u_{k-1}$$
- Hence
$$\sum_{i=0}^{k+1} u_i < u_{k+2} + u_{k-1}$$

which completes the proof. □

Lemma 5.5

Lemma 5.5: There are no singletons or pairs of the u_i with equal sums

Proof:

- There are no equal singletons because $\{u_i\}$ is a strictly increasing sequence.
- Suppose we have two pairs of the u_i with equal sums. In other words suppose we have sets $\{u_{i_1}, u_{i_2}\}$ and $\{u_{j_1}, u_{j_2}\}$ which are disjoint such that

$$u_{i_1} + u_{i_2} = u_{j_1} + u_{j_2}.$$

  Assume without loss of generality that $u_{i_1} < u_{i_2}$, $u_{j_1} < u_{j_2}$.
- Since the sets are disjoint, one of them must contain a largest element (from both sets), so assume without loss of generality that $u_{j_2} > u_{i_2}$.
- Now we know by Lemma 4 that $u_{j_2} > u_{j_2-1} + u_{j_2-2}$
- Since $u_{j_2} > u_{i_2} > u_{i_1}$ we see that $u_{i_2} \leq u_{j_2-1}$ and $u_{i_1} \leq u_{j_2-2}$.
- Hence $u_{j_2} > u_{i_2} + u_{i_1}$ and since $u_{j_1} \geq 0$ we see that we cannot have that $u_{i_1} + u_{i_2} = u_{j_1} + u_{j_2}$ and hence we obtain a contradiction. □

Lemma 6

Lemma 6: There are no distinct triples of the u_i with equal sums

Proof:

- The result will follow (using the same technique used in the previous lemma) if we show that
$$u_{n+1} \geq u_n + u_{n-1} + u_{n-2} \text{ for } n \geq 2$$
- This can be verified by hand from the earlier Table for $2 \leq n \leq 7$ with equality occuring for $3 \leq n \leq 6$.
- *Induction hypothesis:* suppose that $u_{m+1} \geq u_m + u_{m-1} + u_{m-2}$ for $m \geq 2$ holds for $1 \leq m \leq n-1$. We will show it holds for n too.
- For $n > 7$ we have that $r > 3$ so that $n - r < n - 3$.
- By definition $u_{n+1} = 2u_n - u_{n-r}$, hence $u_{n+1} > 2u_n - u_{n-3} = u_n + (u_n - u_{n-3})$.
- By the induction hypothesis we see that $u_n > u_{n-1} + u_{n-2} + u_{n-3}$. Hence
$$u_{n+1} > u_n + u_{n-1} + u_{n-2} + u_{n-3} - u_{n-3} = u_n + u_{n-1} + u_{n-2}$$
as desired.

Proof of Theorem 4

Theorem 4: We have that

$$\lim_{n\to\infty} \frac{u_n}{2^n} = \alpha$$

where $0 < \alpha < \frac{1}{2}$

Remark: In particular, this result implies that the sequence u_n behaves/grows like 2^n.

Proof:

- Define
$$\alpha_n := \frac{u_n}{2^n}.$$
- In the range $\frac{1}{2}m(m+1)+1 \leq n \leq \frac{1}{2}m(m+1)(m+2)$ we have that $r = m+1$.
- Now we know by definition of the Conway-Guy sequence that $u_{n+1} = 2u_n - u_{n-r}$.

Proof of Theorem 4

Proof:

- Thus

$$\alpha_{n+1} = \frac{u_{n+1}}{2^{n+1}} = \frac{2u_n}{2^{n+1}} - \frac{u_{n-r}}{2^{n+1}} = \alpha_n - \frac{u_{n-m-1}}{2^{n+1}} = \alpha_n - \frac{\alpha_{n-m-1}}{2^{m+2}}$$

- If we sum α_{n+1} over the range $\frac{1}{2}m(m+1)+1 \leq n \leq \frac{1}{2}m(m+1)(m+2)$ we get

$$\alpha_{m(m+1)(m+2)/2} = \alpha_{\frac{1}{2}m(m+1)+1} - 2^{-(m+2)} \sum_{n=\frac{1}{2}m(m-1)}^{\frac{1}{2}m(m+1)(m+2)} \alpha_n$$

- If we substitute $m+j-1$ for m and sum the above from $j=1$ to $j=p$, we get

$$\alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - \sum_{j=1}^{p} 2^{-(m+j-1)} \sum_{n=\frac{1}{2}(m+j-1)(m+j-2)}^{\frac{1}{2}(m+j)(m+j+1)} \alpha_n$$

Proof of Theorem 4

Proof:

- Since $\alpha_{23} = 2095003 \times 2^{-23} < \frac{1}{4}$, Lemma 3 implies that

$$\alpha < \alpha_n < \frac{1}{4}$$

  for $n \geq 23$.
- Thus for $m \geq 8$, we have that $(m+j-1)(m+j-2)/2+1 \geq 29$ and thus in the range $\frac{1}{2}(m+j-1)(m+j-2) \leq n \leq \frac{1}{2}(m+j)(m+j+1)$ we see that $\alpha < \alpha_n < \frac{1}{4}$ and hence that

$$\alpha(m+j) < \sum_{n=\frac{1}{2}(m+j-1)(m+j-2)}^{\frac{1}{2}(m+j)(m+j+1)} \alpha_n < \frac{1}{4}(m+j)$$

Proof of Theorem 4

Proof:

- Recall we had that

$$\alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - \sum_{j=1}^{p} 2^{-(m+j-1)} \sum_{n=\frac{1}{2}(m+j-1)(m+j-2)}^{\frac{1}{2}(m+j)(m+j+1)} \alpha_n$$

- Let $T(p) = \sum_{j=1}^{p} 2^{-(m+j-1)} \sum_{n=\frac{1}{2}(m+j-1)(m+j-2)}^{\frac{1}{2}(m+j)(m+j+1)} \alpha_n$ so that

$$\alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - T(p)$$

- Since

$$\alpha(m+j) < \sum_{n=\frac{1}{2}(m+j-1)(m+j-2)}^{\frac{1}{2}(m+j)(m+j+1)} \alpha_n < \frac{1}{4}(m+j)$$

  we have that

$$2^{-(m+p-1)}\alpha(m+p) < T(p) < 2^{-(m+p-1)}\frac{1}{4}(m+p)$$

Proof of Theorem 4

Proof:

- Just through some algebraic manipulations we then have that

$$2^{-m-1}\alpha(m+2-(m+p+2)2^{-p}) < T(p) < 2^{-m-3}(m+2-(m+p+2)2^{-p})$$

- If we keep m fixed and let $p \to \infty$ and $\beta = \lim_{p\to\infty} T(p)$, then we have that

$$2^{-m-1}\alpha(m+2) < \beta < 2^{-m-3}(m+2)$$

- Now recall that

$$\alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - T(p)$$

- So if we keep m fixed and let $p \to \infty$ then

$$\alpha = \lim_{p\to\infty} \alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - \beta$$

  where β lies between $\alpha(m+2)2^{-m-1}$ and $(m+2)2^{-m-3}$.
- Now we have a good bound on α to work with.

Proof of Theorem 4

Proof:

- From the prev. slide we had

$$\alpha = \lim_{p\to\infty} \alpha_{\frac{1}{2}m(m+p)(m+p+1)} = \alpha_{m(m+1)/2+1} - \beta$$

  where β lies between $\alpha(m+2)2^{-m-1}$ and $(m+2)2^{-m-3}$.
- Thus

$$\alpha_{m(m+1)/2+1} - (m+2)2^{-m-3} < \alpha < \alpha_{m(m+1)/2+1} - \alpha(m+2)2^{-m-1}$$

- For $m = 26$, using the fact that $\alpha < \alpha_{m(m+1)/2+1}$ we have

$$\alpha_{352} - 28 \times 2^{-29} < \alpha < \frac{\alpha_{352}}{1 + 28 \times 2^{-27}} < \alpha_{352} - 26 \times 2^{-29}$$

Proof of Theorem 4

Proof:

- We saw that for $m = 26$ we have

$$\alpha_{352} - 28 \times 2^{-29} < \alpha < \frac{\alpha_{352}}{1 + 28 \times 2^{-27}} < \alpha_{352} - 26 \times 2^{-29}$$

- A computer calculation gave

$$\alpha_{352} = 0.235125333862141\ldots$$

- One then gets

$$\alpha = 0.23512524581118\ldots$$

□

Theorem 7

Theorem 7: If $s \geq 0$ and $1 \leq t \leq s+2$, then with the same convention as in Theorem 6,

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Proof:

- The theorem may be checked by hand from Table 1 for $0 \leq s \leq 2$ and $1 \leq t \leq s+2$.
- We claim that if the theorem is true for some value of s and t, it is also true for the same value of s and $t+1$ in place of t.
- Suppose that for s and t we have that

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}$$

Theorem 7

Want to show:

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Proof:

- From Lemma 4 we know that $u_{T_s+t+2} > u_{T_s+t+1} + u_{T_s+t}$. Thus $u_{T_s+t+2} - u_{T_s+t+1} > u_{T_s+t}$.
- Thus we can add $u_{T_s+t+2} - u_{T_s+t+1}$ to the left hand side of the inequality and u_{T_s+t} to the right hand side of the inequality to yield that:

$$\begin{aligned} u_{T_s+t+1} + u_{T_s+t+2} - u_{T_s+t+1} &= u_{T_s+t+2} \\ &> u_{T_s+t} + \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i} \\ &= \sum_{i=0}^{T_s+t} u_i + \sum_{i=2}^{s} u_{T_i} \end{aligned}$$

Theorem 7

Want to show:

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Proof:

- We claim that if the theorem is true for some $s \geq 2$ and $t = 1$, then it is also true for the same value of $s + 1$ and $t = 1$.
- Suppose that for $s \geq 2$ and $t = 1$ we have that

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i} \tag{5}$$

- <u>Claim:</u>

$$u_{T_{s+1}+t+1} - u_{T_s+t+1} > \sum_{i=T_s+t}^{T_{s+1}} u_i + u_{T_{s+1}} \tag{6}$$

Theorem 7

Want to show:

$$u_{T_s+t+1} > \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i}.$$

Proof:

- If we add the left hand side of 6 to the left of 5 and the right hand side of 6 to the right hand side of 5 we get:

$$\begin{aligned}
u_{T_{s+1}+t+1} &= u_{T_{s+1}+t+1} - u_{T_s+t+1} + u_{T_s+t+1} \\
&> \sum_{i=0}^{T_s+t-1} u_i + \sum_{i=2}^{s} u_{T_i} + \sum_{i=T_s+t}^{T_{s+1}} u_i + u_{T_{s+1}} \\
&= \left(\sum_{i=0}^{T_s+t-1} u_i + \sum_{i=T_s+t}^{T_{s+1}} u_i \right) + \left(\sum_{i=2}^{s} u_{T_i} + u_{T_{s+1}} \right) \\
&= \sum_{i=0}^{T_{s+1}} u_i + \sum_{i=2}^{s+1} u_{T_i} + u_{T_{s+1}}
\end{aligned}$$

as desired, provided the claim holds. □

Theorem 9

Theorem 9: Under conditions A to D, u_{T_s+t-1} belongs to the minor set.

Proof:

- From condition A, we have two sets of the u_i with equal sums, those being

$$\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\} \quad \text{and} \quad \{u_{j_1}, \dots, u_{j_l}\}$$

  . We order these sets so that $u_{i_m} < u_{i_{m+1}}$ and $u_{j_m} < u_{j_{m+1}}$ for $1 \leq m \leq k$ and $1 \leq m \leq l$ respectively.
- Suppose without loss of generality that u_{j_l} is the largest element from both sets.
- Rewrite j_l to be $j_l = T_{s+1} + t + 1$ for some s and t so that the largest element from both sets is $u_{T_{s+1}+t+1}$.
- So from the two sets $\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$ and $\{u_{j_1}, \dots, u_{j_l}\}$, the set $\{u_{i_1}, u_{i_2}, \dots, u_{i_k}\}$ is the minor set.

Theorem 9

Theorem 9: Under conditions A to D, u_{T_s+t-1} belongs to the minor set.

Proof contd.:

- Suppose u_{T_s+t-1} does not belong to the minor set, i.e.

$$u_{T_s+t-1} \notin \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}.$$

- Then by Theorem 8 the set $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ must contain the $s+1$ elements, u_i for $T_s + t + 1 \leq i \leq T_{s+1} + t$.
- Thus we have

$$u_{i_1} + u_{i_2} + \cdots + u_{T_s+t+1} + u_{T_s+t+2} + \cdots + u_{T_{s+1}+t-1} + u_{T_{s+1}+t} = u_{j_1} + \cdots + u_{T_{s+1}+t+1}$$

- One can check that $u_{T_{s+1}+t+1} = 2u_{T_{s+1}+t} - u_{T_s+t-1}$ (this follows just from the definition of the Conway-Guy sequence)
- We substitute $u_{T_{s+1}+t+1} = 2u_{T_{s+1}+t} - u_{T_s+t-1}$ in the equality from the previous slide to get

$$\begin{aligned} &u_{i_1} + u_{i_2} + \cdots + u_{T_s+t+1} + u_{T_s+t+2} + \cdots + u_{T_{s+1}+t-1} + u_{T_{s+1}+t} \\ &= u_{j_1} + \cdots + 2u_{T_{s+1}+t} - u_{T_s+t-1} \end{aligned}$$

Theorem 9

Proof continued:

- On the previous slide we arrived at the following equation:

$$u_{i_1} + u_{i_2} + \cdots + u_{T_s+t+1} + u_{T_s+t+2} + \cdots + u_{T_{s+1}+t-1} + u_{T_{s+1}+t}$$
$$= u_{j_1} + \cdots + 2u_{T_{s+1}+t} - u_{T_s+t-1}$$

- Then we cancel out a $u_{T_{s+1}+t}$ from either side to get that

$$u_{i_1} + u_{i_2} + \cdots + u_{T_s+t+1} + u_{T_s+t+2} + \cdots + u_{T_{s+1}+t-1} = u_{j_1} + \cdots + u_{T_{s+1}+t} - u_{T_s+t-1}$$

- Now we add a u_{T_s+t-1} to either side to yield that

$$u_{i_1} + u_{i_2} + \cdots + u_{T_s+t+1} + u_{T_s+t+2} + \cdots + u_{T_{s+1}+t-1} + u_{T_s+t-1} = u_{j_1} + \cdots + u_{T_{s+1}+t}$$

- But now the sets $\{u_{i_1}, u_{i_2}, \cdots, u_{T_s+t+1}, u_{T_s+t+2}, \cdots, u_{T_{s+1}+t-1}, u_{T_s+t-1}\}$ and $\{u_{j_1}, u_{j_2}, \cdots, u_{T_{s+1}+t}\}$ have equal sums but a smaller largest member, that being $u_{T_{s+1}+t}$.
- This contradicts condition C and the result follows. $\square$

Theorem 10

Theorem 10: Under conditions A to D, the minor set does not contain all the $s+4$ members u_i for $T_s + t - 2 \leq i \leq T_{s+1} + t$.

Proof:

- If the minor set contained these $s+4$ members, it's sum S_1 would be at least

$$\sum_{i=T_s+t-2}^{T_{s+1}+t} u_i$$

- Now we can rewrite the above as

$$\sum_{i=T_s+t-2}^{T_{s+1}+t} u_i = u_{T_s+t-2} + u_{T_s+t-1} + \sum_{i=T_s+t}^{T_{s+1}+t} u_i$$

- Theorem 6 says that $\sum_{i=T_s+t}^{T_{s+1}+t} u_i = u_{T_{s+1}+t+1} + \sum_{i=0}^{s} u_{T_i}$ hence

$$S_1 \geq u_{T_s+t-2} + u_{T_s+t-1} + \sum_{i=0}^{s} u_{T_i} + u_{T_{s+1}+t+1}.$$

Theorem 10

Proof continued:

- On the other hand, the sum of the major set, S_2 would be at most

$$u_{T_{s+1}+t+1} + \sum_{i=0}^{T_s+t-3} u_i$$

- This is because, by condition A, the major set cannot contain any of the elements u_i for $T_s + t - 2 \leq i \leq T_{s+1} + t$
- Lemma 5 says that $\sum_{i=0}^{T_s+t-3} u_i \leq u_{T_s+t-2} + u_{T_s+t-5}$
- This implies that $S_2 \leq u_{T_{s+1}+t+1} + u_{T_s+t-2} + u_{T_s+t-5}$
- We thus have the following situation:

$$u_{T_s+t-2}+u_{T_s+t-1}+\sum_{i=0}^{s} u_{T_i}+u_{T_{s+1}+t+1} \leq S_1 = S_2 \leq u_{T_{s+1}+t+1}+u_{T_s+t-2}+u_{T_s+t-5}$$

- This implies that

$$u_{T_s+t-2} + u_{T_s+t-1} + \sum_{i=0}^{s} u_{T_i} + u_{T_{s+1}+t+1} \leq u_{T_{s+1}+t+1} + u_{T_s+t-2} + u_{T_s+t-5}$$

Theorem 10

Proof continued:

- Which implies that

$$u_{T_s+t-1} + \sum_{i=0}^{s} u_{T_i} \leq u_{T_s+t-5}$$

- This is a contradiction since $u_{T_s+t-1} > u_{T_s+t-5}$ because the $\{u_i\}$ is a strictly increasing sequence.
- This completes the proof. $\square$

Lemma 8

Lemma 8: If $s \geq 0$, with the convention of Theorem 6,

$$\sum_{i=2}^{s} u_{T_i} < \frac{1}{2}\left(u_{T_s+1} + u_{T_{s-1}+2}\right)$$

Proof:

- If $s = 0$, then $-1 < \frac{1}{2}(1+2)$
- If $s = 1$, then $0 < \frac{1}{2}(2+2)$
- If $s = 2$, then $4 < \frac{1}{2}(7+4)$
- If $s = 3$, then $4 + 24 < \frac{1}{2}(44+13)$

Lemma 8

Proof:

- Assume the theorem is true for $s = v \geq 3$, we show it is true for $v + 1$.
- Then

$$\begin{aligned}
\sum_{i=0}^{v+1} u_{T_i} &= u_{T_{v+1}} + \sum_{i=0}^{v} u_{T_i} \\
&< u_{T_{v+1}} + \frac{1}{2}(u_{T_v+1} + u_{T_{v-1}+2}) \quad \text{by induction hypothesis} \\
&= \frac{1}{2}(2u_{T_{v+1}} + u_{T_v+1} + u_{T_{v-1}+2}) \\
&= \frac{1}{2}\left((2u_{T_{v+1}} - u_{T_v}) + u_{T_v} + u_{T_v+1} + u_{T_{v-1}+2}\right) \\
&= \frac{1}{2}\left(u_{T_{v+1}+1} + u_{T_v} + u_{T_v+1} + u_{T_{v-1}+2}\right) \quad \text{since } 2u_{T_{v+1}} - u_{T_v} = u_{T_{v+1}+1} \\
&\leq \frac{1}{2}\left(u_{T_{v+1}+1} + u_{T_v} + u_{T_v+1} + u_{T_v-1}\right)
\end{aligned}$$

since $T_{v-1} + 2 \leq T_v - 1$ holds for $n \geq 3$ so that $u_{T_{v-1}+2} \leq u_{T_v-1}$

$$\leq \frac{1}{2}\left(u_{T_{v+1}+1} + u_{T_v+2}\right) \quad \text{since } u_{T_v+2} \geq u_{T_v} + u_{T_v+1} + u_{T_v-1}. \square$$

Lemma 9

Lemma 9: If $v > T_{s+1}$, then $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof:

- The case $s = 0$ is just Lemma 1, since it boils down to saying that $u_v < u_{v+1}$
- The case $s = 1$ is just Lemma 4, since it just says that $u_{v-1} + u_v < u_{v+1}$
- The case $s = 2$ just says that $u_{v-2} + u_{v-1} + u_v < u_{v+1}$ and this is true by inequality (12) in the paper
- The case $s = 3$, just says that $u_{v-3} + u_{v-2} + u_{v-1} + u_v < u_{v+1}$ and this is true by inequality (13) in the paper.

Lemma 9

Lemma 9: If $v > T_{s+1}$, then $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof continued:

- We now handle the case that $v = T_{s+1} + 1$.
- Note firstly that $v - s = T_{s+1} + 1 - s$. We saw earlier that $T_{s+1} + 1 = T_s + s + 2$ which implies that $v - s = T_s + 2$.
- Hence

$$\sum_{i=v-s}^{v} u_i = \sum_{T_s+2}^{T_{s+1}+1} u_i$$

- Through simple algebra we see that

$$\sum_{T_s+2}^{T_{s+1}+1} u_i = \sum_{T_s+1}^{T_{s+1}+1} u_i - u_{T_s+1}$$

- Then using Theorem 6 with $t = 1$ implies that

$$\sum_{T_s+1}^{T_{s+1}+1} u_i - u_{T_s+1} = u_{T_{s+1}+2} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+1}$$

Lemma 9

Lemma 9: If $v > T_{s+1}$, then $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof continued:

- On the previous slide we arrived at

$$\sum_{T_s+1}^{T_{s+1}+1} u_i - u_{T_s+1} = u_{T_{s+1}+2} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+1}.$$

- Then using Lemma 8 on $\sum_{i=2}^{s} u_{T_i}$ in the above we see that

$$u_{T_{s+1}+2} + \sum_{i=2}^{s} u_{T_i} - u_{T_s+1} < u_{T_{s+1}+2} + \frac{1}{2}\left(u_{T_s+1} + u_{T_{s-1}+2}\right) - u_{T_s+1}$$

- Then provided $T_{s-1} + 2 \leq T_s + 1$ (which is true for $s \geq 1$) we see that $u_{T_s+1} \geq u_{T_{s-1}+2}$
- This implies that

$$u_{T_{s+1}+2} + \frac{1}{2}\left(u_{T_s+1} + u_{T_{s-1}+2}\right) - u_{T_s+1} \leq u_{T_{s+1}+2}.$$

Lemma 9

Lemma 9: If $v > T_{s+1}$, then $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof continued:

- Putting all this together we see that

$$\sum_{i=v-s}^{v} u_i \leq u_{T_{s+1}+2}.$$

- Now suppose the result holds for $v = w > T_{s+1}$.
- Through simple algebra we get that

$$\sum_{i=w+1-s}^{w+1} u_i = u_{w+1} - u_{w-s} + \sum_{i=w-s}^{w} u_i$$

- Then since the result holds for w, we see that $\sum_{i=w-s}^{w} u_i < u_{w+1}$ and hence that

$$u_{w+1} - u_{w-s} + \sum_{i=w-s}^{w} u_i < 2u_{w+1} - u_{w-s}$$

Lemma 9

Lemma 9: If $v > T_{s+1}$, then $\sum_{i=v-s}^{v} u_i < u_{v+1}$.

Proof continued:

- Recall that
$$u_{w+1} - u_{w-s} + \sum_{i=w-s}^{w} u_i < 2u_{w+1} - u_{w-s}$$
- Recall the defining property of the sequence of the u_i, that being that
$$u_{n+1} = 2u_n - u_{n-r}$$
for $n \geq 1$ and $r = \langle\sqrt{2n}\rangle$.
- Recall that we assumes that $w > T_{s+1} = \frac{1}{2}(s+1)(s+2)$. Hence $w - s > T_{s+1} - s = T_s + 2$ (since we have the identity that $T_{s+1} + 1 = T_s + s + 2$)
- This shows that $w - s > w + 1 - r$ (here we take $w + 1$ in place of n which yields a value for r) which implies that $u_{w-s} > u_{w+1-r}$ since the u_i are monotonically increasing
- This then shows that $2u_{w+1} - u_{w-s} < 2u_{w+1} - u_{w+1-r} = u_{w+2}$ using the defining property of the sequence of the u_i with $w + 1$ in place of n $\square$