Compressive Sensing:

Concepts & Applications

27 Colóquio Brasileiro de Matemática

People / Motivation

- Instructors
  - Luiz Velho (IMPA)
  - Eduardo Silva (UFRJ)
  - Adriana Schulz (IMPA)
- Inside Stories
  - Curves and Surfaces 2006, Avignon
  - Final Project, COPPE / UFRJ

Course Outline

- Introduction / Classical Sampling Theory
- Compression / Representation Theory
- The Compressive Sensing Framework
- Quantization / Image Processing
- Applications in Graphics

New Topic!

- Communications of ACM, May 2009
  - Rethinking Signal Processing

> "The theory was so revolutionary when it was created a few years ago that an early paper outlining it was initially rejected on the basis that its claims appeared impossible to substantiate."

Compressed sensing is leading to new ways of looking at math problems in seemingly unrelated areas.

Compressed sensing has applications for biomedical imaging, digital photography, and other forms of analog-to-digital conversion.

Change of Paradigm

- Sampling / Reconstruction

~ 55 years!

- Sensing / Recovery

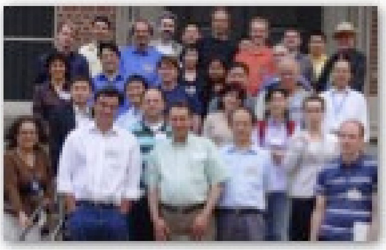
Who is Who?

- Emmanuel Candes
- David Donoho
- Richard Baraniuk

▶ *Many others....*

Historical Events

- IMA - U. Minnesota, New Directions Short Course: Compressive Sampling and Frontiers in Signal Processing (June, 2007)
- Duke University Compressive Sensing Workshop (February, 2009)



Resources

- Rice CS Resources

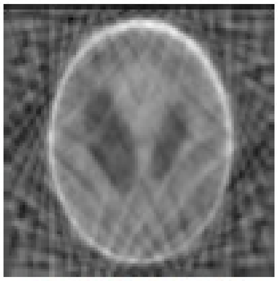
  (http://www.compressedsensing.com/)

- L1-Magic

  (http://www.l1-magic.org/)

Where it all begun?

- The "*Phantom*" Case: Experimental Mathematics
  - MRI Reconstruction from 5% samples in K-space



back-projection

total variation (exact)

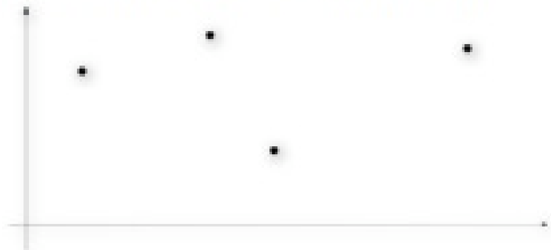
"That's where the surprise came in. What I was not expecting that it would give me the truth! It was the birth of compressed sensing" - E. Candes (CACM 09)

Sampling & Reconstruction

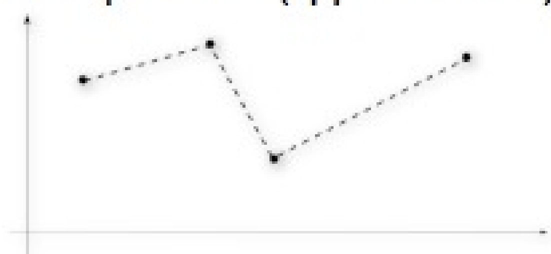
- Scattered Data Interpolation
- Signal Representation
- Shannon's Sampling Theorem
- Characteristics and Properties
- Examples

Scattered Data Interpolation

- Data Samples (*non-uniform*)



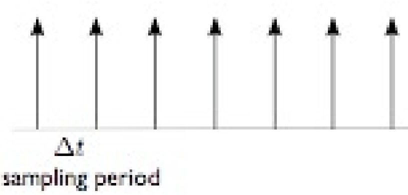
- PL Interpolation (*approximation*)



Signal Representation (I)

- Uniform Point Sampling
  - Impulse Train

$$L_\Delta = \{i\Delta t \mid i \in \mathbb{Z}\}$$



  - Function Sampling

$$f_\Delta(t) = \sum_{n=-\infty}^{\infty} f(n\Delta t)\, \delta(t - n\Delta t)$$

  - Sample Sequence

$$f_\Delta = \{\ldots, f(\Delta t), f(2\Delta t), \ldots\}$$

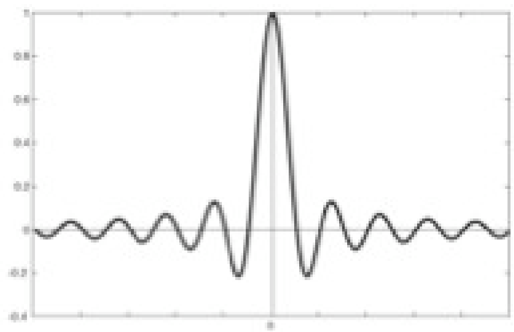
Signal Representation (II)

- Ideal Reconstruction

$$f(t) = \sum_{k=-\infty}^{+\infty} 2\Omega\Delta t\ f(k\Delta t)\ \mathrm{sinc}(2\pi\Omega(t - k\Delta t))$$

- Shannon Basis
  - sinc Function

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$



Sampling Theorem

- (Shannon-Whittaker, 1949)

Let f be a bandlimited signal with $\hat{f} \subset [-\Omega, \Omega]$. Then, f can be exactly reconstructed from the uniform sample sequence $\{f(m\Delta t) \mid m \in \mathbb{Z}\}$ by interpolation functions $\{\ \mathrm{sinc}(t - m\Delta t)\ \}$, if $\Delta t \leq 1/(2\Omega)$.

Key Concepts

- Bandlimited Signal

$$\hat{f} \subset [-\Omega, \Omega]$$

(i.e., no frequencies above Ω)

- Sampling Frequency

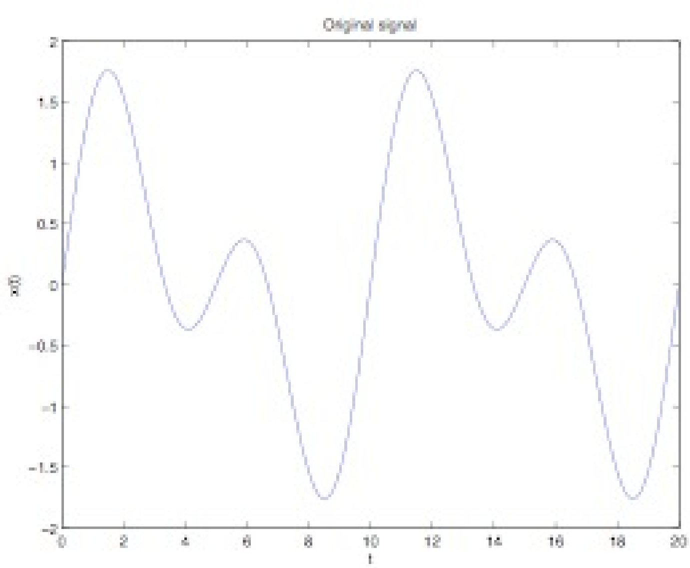
$$\omega = \frac{2\pi}{\Delta t}$$

- Nyquist Limit

$$\Delta t < \frac{1}{2\Omega}$$

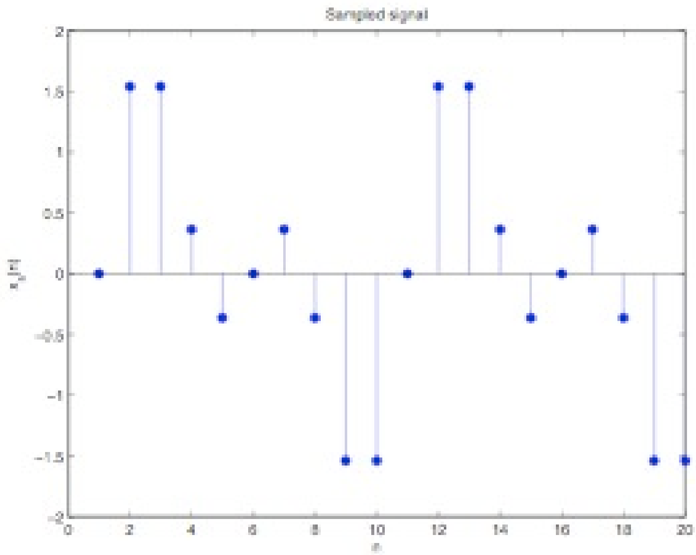
Example 1

- Signal: f(t)



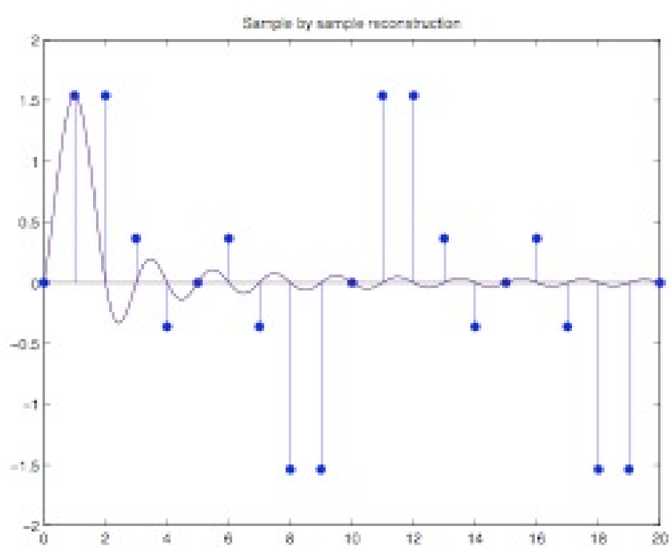
Example 1

- Uniform Sampling: $\{f(t_i)\}$



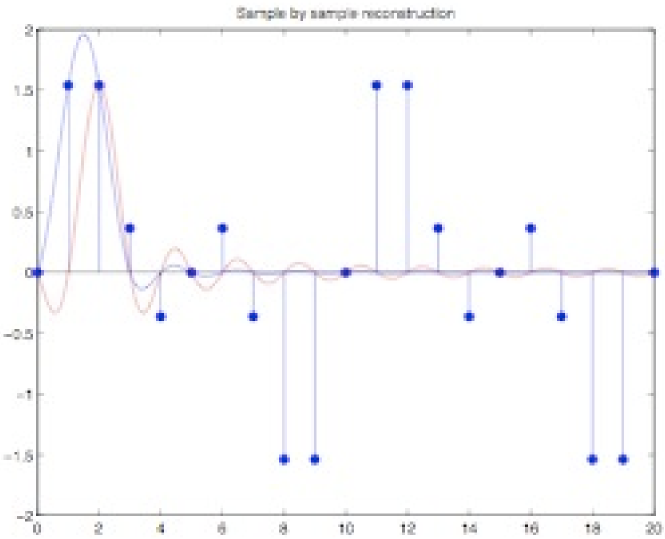
Example 1

- Reconstruction: step 1



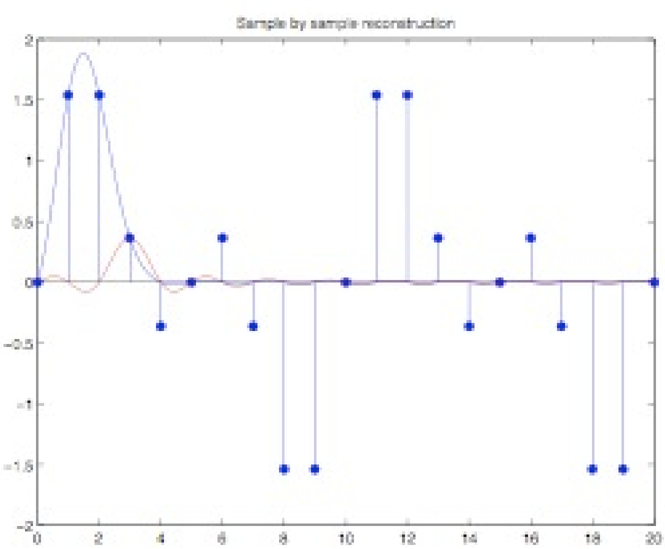
Example 1

- Reconstruction: step 2



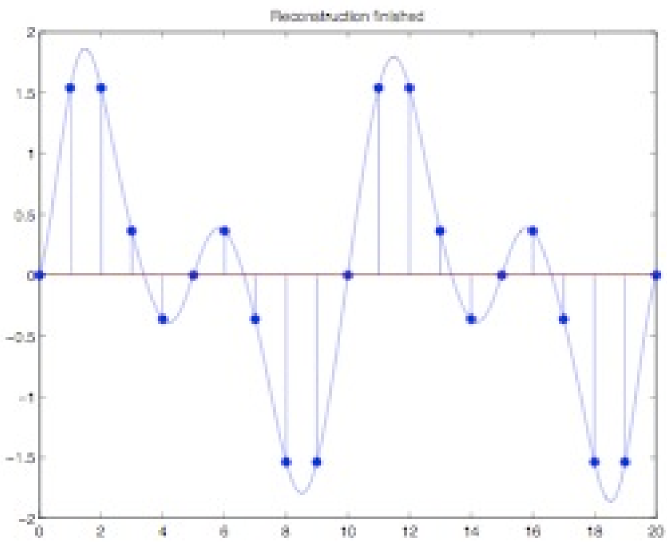
Example 1

- Reconstruction: step 3



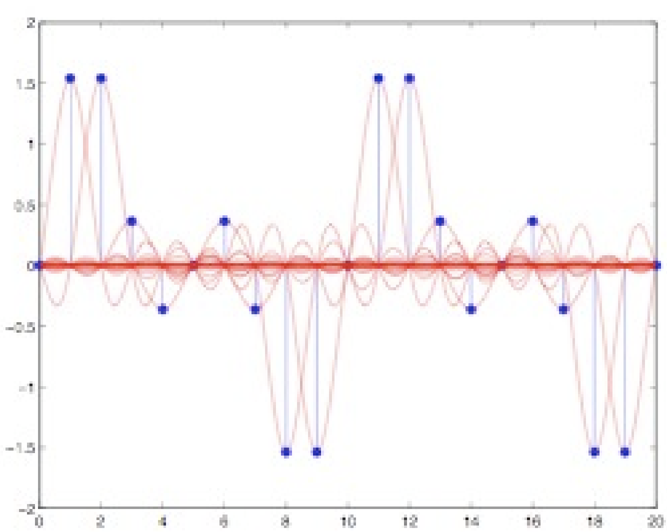
Example 1

- Reconstruction: complete



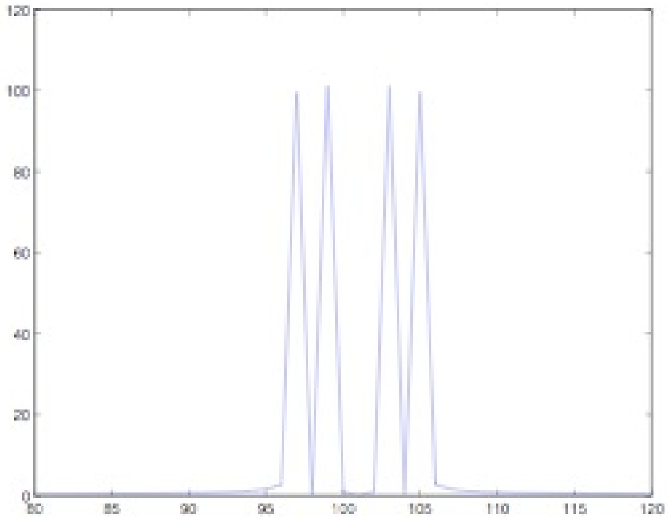
Example 1

- Shannon Interpolation Basis

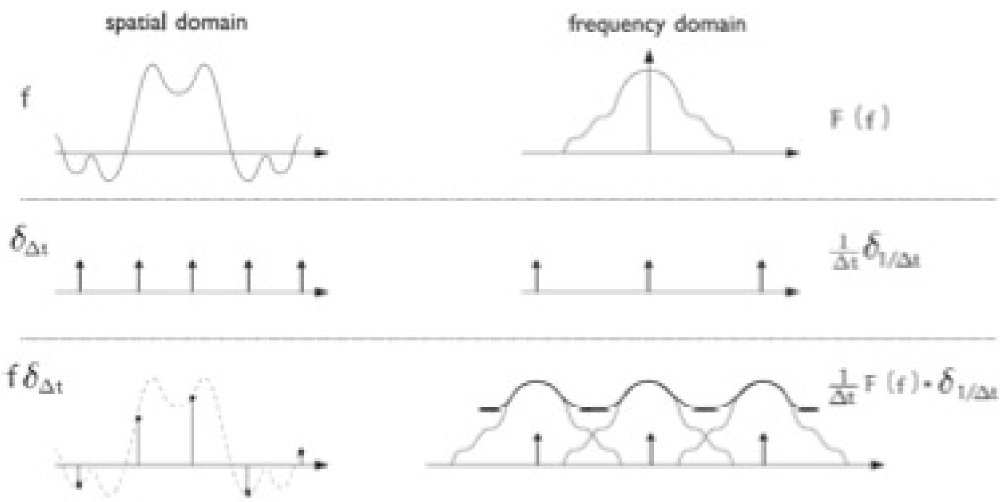


Example 1

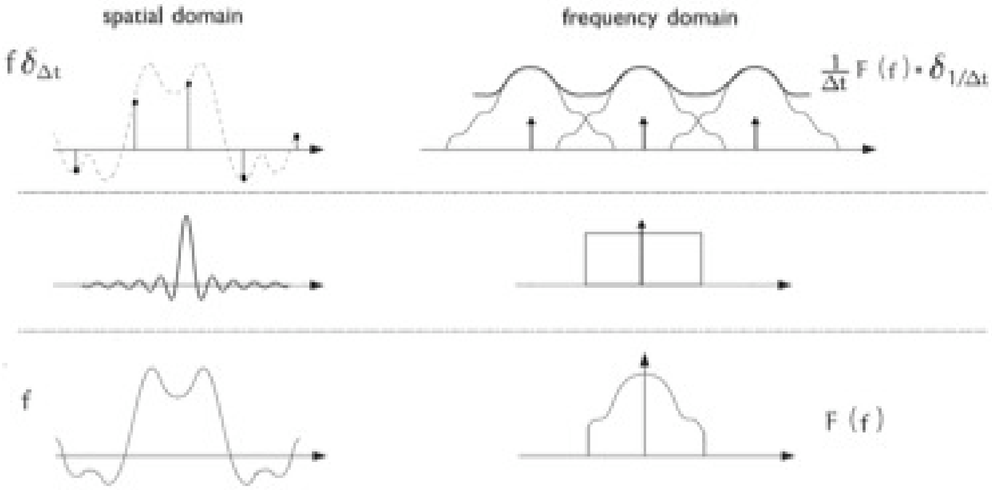
- Fourier Transform: $\mathcal{F}(w)$



What is going on ? (1)

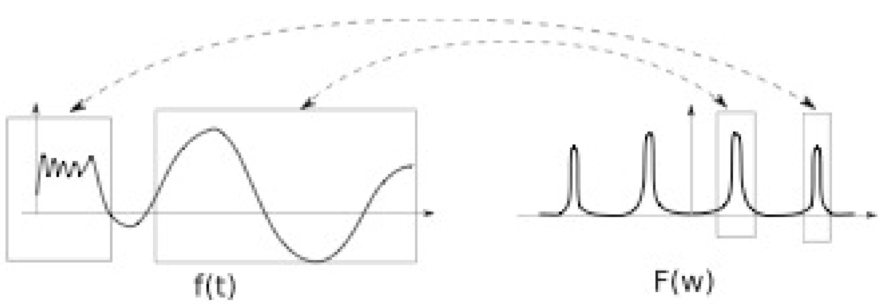


What is going on ? (II)



Shannon was a Pessimist!

- Nyquist Limit = worst case bound:
  - Highest Frequency
- Global vs. Local



CS to the Rescue :)

- New Paradigm: *Sensing & Recovery*
  - First CS Theorem
  - Basic Framework
  - Toy Example
  - Key Notions
  - CS Fourier Sampling
  - Back to Phantom

First CS Theorem

- Theorem (Candes, Romberg and Tao, 2004)

Supose $f \in R^N$ is K-sparse, sample $\hat{f}$
at M random frequencies, $\omega_1, \ldots \omega_m$

$$M \gtrsim K \log N$$

then, solving

$$\min ||f||_{\ell_1} \quad \text{s.t} \quad \hat{f}(\omega_m) = y_m$$

reconstructs f with overwhelming probability.

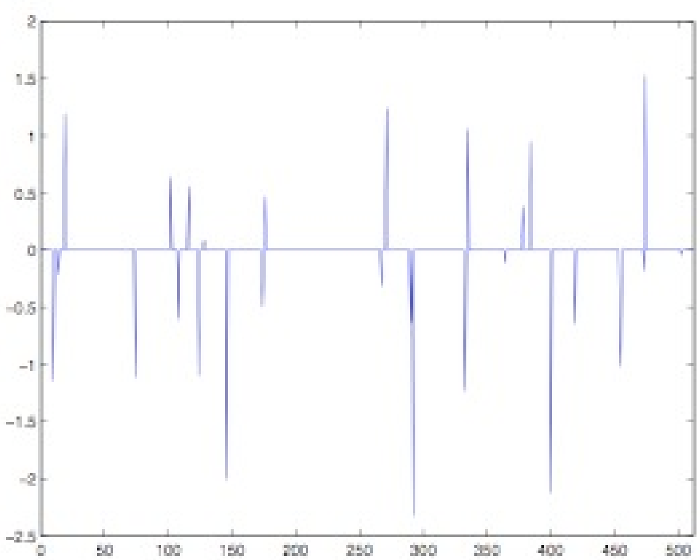
Matlab

- nonlinear_sampling.m (E. Candes)

```
%% Prepare data
n = 512;                % Size of the signal
m = 64;                 % Number of samples (undersample by a factor 8)
%
k = 0:n-1; t = 0:n-1;
F = exp(-i*2*pi*k'*t/n)/sqrt(n);          % Fourier matrix
freq = randsample(n,m);
A = [real(F(freq,:)); imag(F(freq,:))]; % Incomplete Fourier matrix
S = 28;
support = randsample(n,S);
x0 = zeros(n,1); x0(support) = randn(S,1);
b = A*x0;
%% Solve l1
cvx_begin
  variable x(n);
  minimize(norm(x,1));
  subject to
      A*x == b;
cvx_end
```

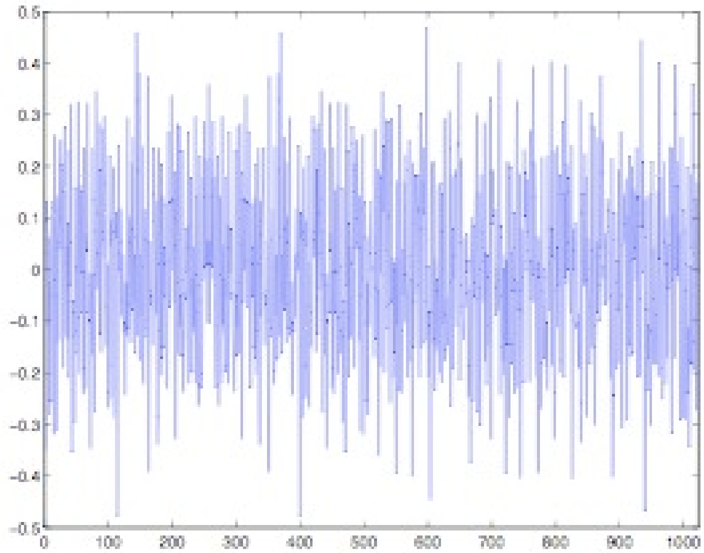
Example 2

- Sparse *f* in Time Domain (K=28)



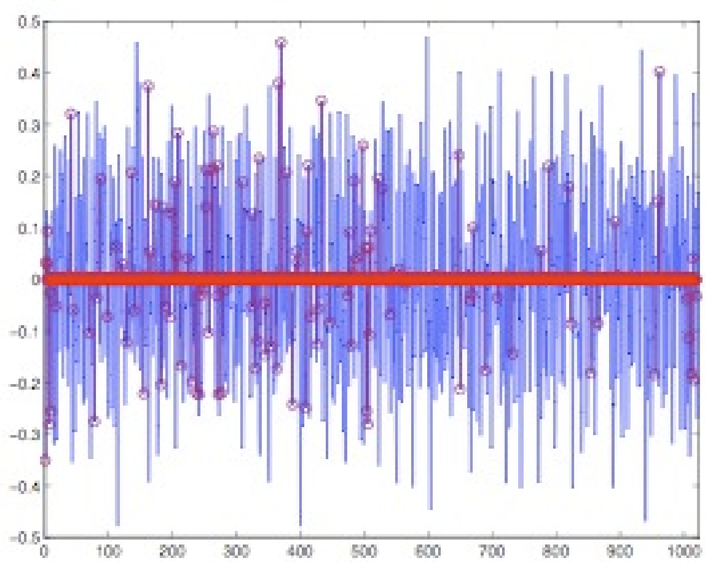
Example 2

- Fourier Transform of f (dense)



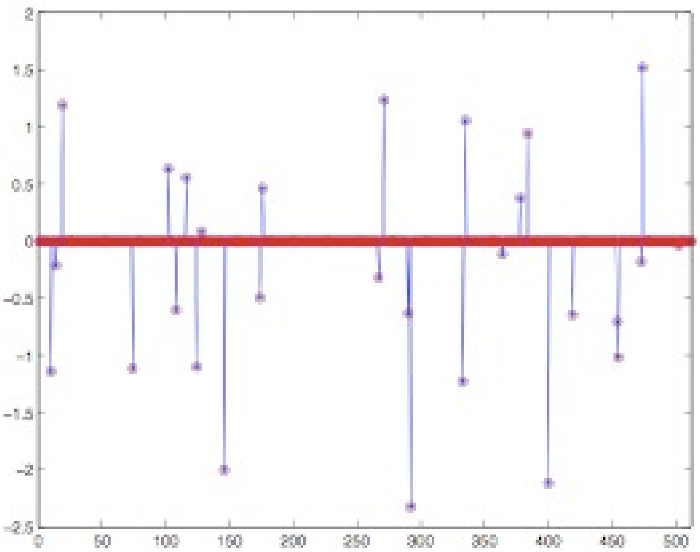
Example 2

- Random Fourier Sampling of M = 64 frequencies



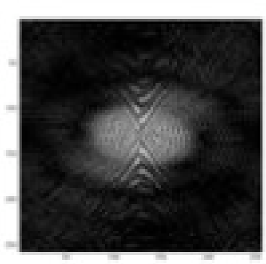
Example 2

- ℓ_1 Reconstruction (*Perfect Recovery!*)

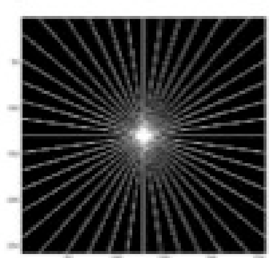


Back to Phantom

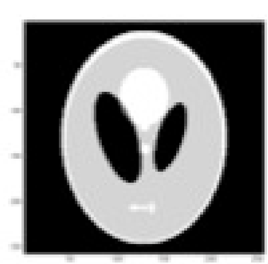
- MRI - Connection with Fourier Data and ℓ_1
  - *k-space* measures / inverse problem



Fourier Transform



Sampling Pattern



TV Reconstruction

Simple CS

- Theorem:

  Given a k-sparse signal X of size N,
  then X can be recovered with overwhelming probability by sensing it M times,
  with $M \ll N$.

Basic Framework

- Sensing

$$y_i = < x, \phi_i >$$

  - Test Functions

$$\phi_i \quad i = 1, \ldots, M$$

- Recovery
  - Optimization

$$\min_x ||x|| \qquad \text{s.t.} \quad y = \Phi x$$

Toy Example

- K = 1
- N = 16
- M = 7
- ϕ = Bernoulli
  - P(-1) = 0.5
  - P(+1) = 0.5

Matlab

- simple_cs.m

```
% sparsity
k = 1;
n = 16;
m = 7;
% test functions
phi = sign(rand(m,n) - 0.5);
% data
i = 9;
f = 8;
x = zeros(n,1);
x(i) = f;
% probing
y = phi * x;
% optimization (exhaustive search)
t = eye(n,n) * f;
z = phi * t;
```

Finding the Answer

- Exhaustive Search

```
y =

     8
     8
    -8
    -8
     8
    -8
     8

z =

     8    -8     8    -8     8     8     8     8     8    -8    -8     8    -8     8    -8    -8
    -8    -8    -8    -8     8     8     8     8     8     8    -8    -8    -8    -8     8     8
     8    -8    -8    -8     8    -8     8     8    -8     8    -8     8     8    -8     8     8
     8     8    -8     8     8     8     8     8    -8    -8    -8     8     8     8    -8     8
     8     8     8     8    -8    -8    -8     8     8     8    -8    -8    -8    -8     8     8
     8     8    -8     8     8    -8     8     8    -8    -8    -8     8    -8    -8     8     8
     8    -8    -8     8     8     8    -8     8     8    -8    -8    -8     8    -8     8     8
```

Rules of the Game

- *What we know ?*
  - *k*, Number of Samples ≠ 0 (*i.e., Sparseness*)
  - ϕ, Test Functions (*i.e., Probing Mechanism*)
- *What we don't know ?*
  - f, Signal (*i.e., Location / Value of Significant Samples*)
- *How to determine M ?*
  - Depends on *k* and ϕ

A Bit of Intuition...

- *Why does it work?*
  (i.e., uniqueness of solution)
  - Probability of $e_i\Phi = e_j\Phi \qquad \forall\, i \neq j$
- Questions:
  - Size M ?
    - Sparseness
  - Which Φ ?
    - Incoherency

Fundamental Premises

- Sparseness
  - Compressible Representation
- Incoherency
  - Uncorrelated Measurements
- ℓ_1 Optimization
  - Inverse Problem of Estimation

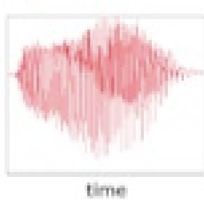
Sparseness

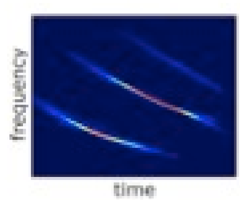
- Signal is sparse if it is concentrated on a "small" set of variables:
  - # Degrees of Freedom << Dimension of Signal
- Many Signals have a Sparse Representation
  - Sparseness Domain

$$x = \Psi z$$

Sparse Representations

- Audio Signals
  - Local Cosines





- Images
  - Wavelets

more on tuesday!

Incoherency

- Representation and Sensing
  - Sparsity Basis ψ_j
  - Measurement Basis ϕ_k

- Coherence between Basis

$$\mu(\Phi, \Psi) = \sqrt{n} \cdot \max_{k,j} | < \phi_k, \psi_j > |$$

Uncertainty Relation

- Uniform Uncertainty Principle (UUP)
  - If x is sparse, then Φ must be dense

$$C_1 \frac{M}{N} \leq \frac{||\Phi x||_2^2}{||x||_2^2} \leq C_2 \frac{M}{N}$$

- Restricted Isometry Property (RIP)
  - Any choice of k { ϕ_i } must form a basis

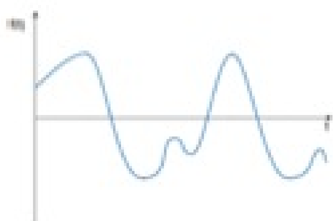
$$1 - \epsilon_k \leq \frac{||\Phi x||_2^2}{||x||_2^2} \leq 1 + \epsilon_k$$

- OBS: M and ϵ_k are related to $\mu(\Phi, \Psi)$

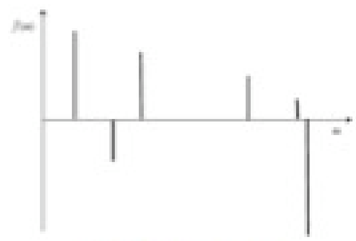
more on wednesday!

Back to Fourier

- Sparse Signals in Frequency Domain
  - Smooth Signals: Low Frequencies
  - Oscillating Patterns: High Frequencies
- Remember Example (1)



space



frequency

CS Fourier Sampling

- Theorem

Supose $\hat{f} \in \mathbb{C}^N$ is K-sparse, sample f at M random time locations, $t_1, \ldots t_m$

$$M \gtrsim K \log N$$

then, solving

$$\min ||\hat{g}||_{\ell_1} \quad \text{s.t} \quad g(t_m) = f(t_m)$$

reconstructs $\hat{f}$ with overwhelming probability.

▸ *OBS: Switch Time and Frequency !*

Comparison

$\hat{f} \in \mathbb{C}^N$ support on Ω in Fourier domain

- Shannon-Whittaker
  - Ω is a known and connected set of size K
  - exact reconstruction from c K uniform samples in time
  - linear interpolation with *sinc* function
- Candes-Romberg-Tao
  - Ω is unknown and arbitrary set of size K
  - exact recovery from ~ K log N random samples
  - nonlinear estimation by optimization

General CS

- Random Sensing Acquisition Theorem:
  - Signal $f \in \mathbb{R}^N$ is K-sparse in Ψ domain
  - Take

    $$M \gtrsim K \cdot \log N$$

    measurements

    $$y_1 =< f, \phi_1 >, \ldots, y_M =< f, \phi_M >$$

    ϕ_m = random incoherent waveform
  - Then solving

    $$\min_x ||x||_{\ell_1} \quad \text{s.t.} \quad \Phi\Psi x = y$$

    recovers f exactly

Characteristics

- Recovery from few non-adaptive measurements
- Simple acquisition followed by optimization
- Sense and compress simultaneously

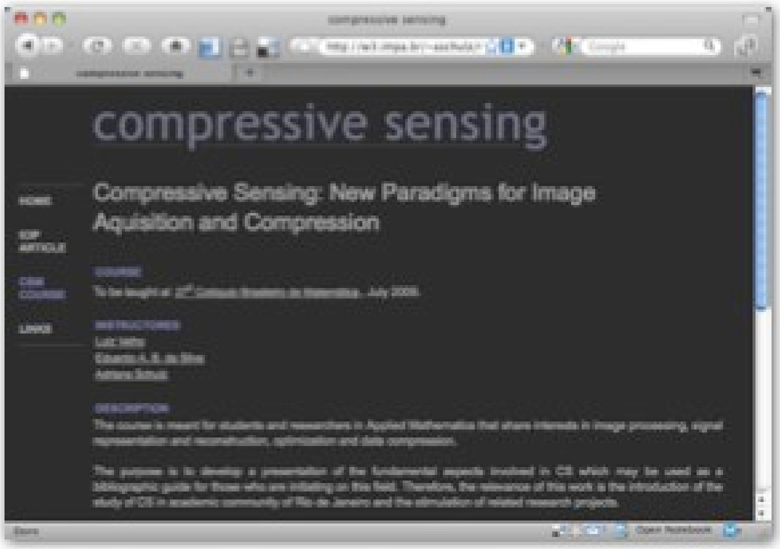
▶ *Asymmetric Process*

What Next ?

- Tuesday (Eduardo)
  - Representation Theory & Compression
- Wednesday (Adriana)
  - CS Theory & Construction of Sensing Ensembles
- Thursday (Adriana)
  - Image Processing & Quantization
- Friday (Luiz)
  - 1 Pixel Camera & Dual Photography

Website

- http://w3.impa.br/~aschulz/CS/



Thanks!