Sobolev Spaces

We plan to give a brief introduction to the classical Sobolev spaces $H^s(\mathbb{R}^n)$. Sobolev spaces measure the differentiability (or regularity) of functions in $L^2(\mathbb{R}^n)$ and they are a fundamental tool in the study of partial differential equations.

Basics

We begin by defining Sobolev spaces.

Definition 1. *Let* $s \in \mathbb{R}$ *. We define the Sobolev space of order* s*, denoted by* $H^s(\mathbb{R}^n)$*, as*

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s f(x) = ((1+|\xi|^2)^{s/2} \widehat{f}(\xi))^\vee(x) \in L^2(\mathbb{R}^n) \right\},$$

with norm $\|\cdot\|_{s,2}$ *defined as*

$$\|f\|_{s,2} = \|\Lambda^s f\|_2.$$

Example 1. *Let* $n = 1$ *and* $f(x) = \chi_{[-1,1]}(x)$. *We have that* $\widehat{f}(\xi) = \sin(2\pi\xi)/(\pi\xi)$.

$$\begin{aligned}
\|f\|_{s,2}^2 = \|\Lambda^s f\|_2^2 &= \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{f}(\xi)|^2 \, dx \\
&= \int_{\mathbb{R}} (1+\xi^2)^s \left|\frac{\sin(2\pi\xi)}{\pi\xi}\right|^2 d\xi \\
&\lesssim \int_{\mathbb{R}} (1+\xi^2)^{s-1} \, d\xi.
\end{aligned}$$

Thus $f \in H^s(\mathbb{R})$ *if* $s < 1/2$.

Example 2. *Let* $n = 1$ *and* $g(x) = \chi_{[-1,1]} * \chi_{[-1,1]}(x)$. *We saw that*

$$\widehat{g}(\xi) = \frac{\sin^2(2\pi\,\xi)}{(\pi\,\xi)^2}.$$

$$\begin{aligned}\|g\|_{s,2}^2 = \|\Lambda^s g\|_2^2 &= \int_{\mathbb{R}} (1+\xi^2)^s |\widehat{g}(\xi)|^2\, dx \\ &= \int_{\mathbb{R}} (1+\xi^2)^s \left|\frac{\sin(2\pi\xi)}{\pi\xi}\right|^4 d\xi \\ &\lesssim \int_{\mathbb{R}} (1+\xi^2)^{s-2}\, d\xi.\end{aligned}$$

Thus $g \in H^s(\mathbb{R})$ *whenever* $s < 3/2$.

Example 3. *Let* $n \geq 1$ *and* $h(x) = e^{-2\pi|x|}$. *In a previous example we saw that*

$$\widehat{h}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{1}{(1+|\xi|^2)^{(n+1)/2}}. \tag{0.1}$$

$$\begin{aligned} \|h\|_{s,2}^2 = \|\Lambda^s h\|_2^2 &= \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{h}(\xi)|^2 \, dx \\ &= c_n \int_{\mathbb{R}^n} (1+|\xi|^2)^s \frac{1}{(1+|\xi|^2)^{n+1}} \, d\xi \\ &= c_n \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-n-1} \, d\xi. \qquad (r^{2s-2n-2} r^{n-1}) \end{aligned}$$

Using polar coordinates we see that $h \in H^s(\mathbb{R}^n)$ *if* $s < n/2 + 1$. *Notice that in this case* s *depends on the dimension.*

Example 4. *Let* $n \geq 1$ *and* $f(x) = \delta_0(x)$. *We already know* $\widehat{\delta}_0(\xi) = 1$. *Then*

$$\|\delta\|_{s,2}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{\delta}(\xi)|^2 \, dx = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \, d\xi \qquad (r^{2s} r^{n-1})$$

Thus $\delta_0 \in H^s(\mathbb{R}^n)$ *if* $s < -n/2$.

From the definition of Sobolev spaces we deduce the following properties.

Proposition 1.

1. *If* $s < s'$, *then* $H^{s'}(\mathbb{R}^n) \subsetneq H^s(\mathbb{R}^n)$.
2. $H^s(\mathbb{R}^n)$ *is a Hilbert space with respect to the inner product* $\langle \cdot, \cdot \rangle_s$ *defined as follows:*

$$\textit{If} \;\; f, g \in H^s(\mathbb{R}^n), \quad \textit{then} \;\; \langle f \, , \, g \rangle_s \; = \int_{\mathbb{R}^n} \Lambda^s f(\xi) \, \overline{\Lambda^s g(\xi)} \, d\xi.$$

We can see, via the Fourier transform, that $H^s(\mathbb{R}^n)$ *is equal to*

$$L^2(\mathbb{R}^n; (1 + |\xi|^2)^s \, d\xi).$$

3. For any $s \in \mathbb{R}$, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

4. If $s_1 \le s \le s_2$, with $s = \theta s_1 + (1-\theta)s_2$, $0 \le \theta \le 1$, then

$$\|f\|_{s,2} \ \le \ \|f\|_{s_1,2}^{\theta}\|f\|_{s_2,2}^{1-\theta}.$$

Proof. 1. Let $f \in H^{s'}(\mathbb{R}^n)$, we show that $f \in H^s(\mathbb{R}^n)$, $s' \ge s$. Then

$$\begin{aligned}\|f\|_{s,2}^2 &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{s-s'}(1+|\xi|^2)^{s'}|\widehat{f}(\xi)|^2\,d\xi \\ &\le \sup_{\mathbb{R}^n}(1+|\xi|^2)^{s-s'}\int_{\mathbb{R}^n}(1+|\xi|^2)^{s'}|\widehat{f}(\xi)|^2\,d\xi \le \|f\|_{s',2}^2.\end{aligned}$$

□

4. Let $s = \theta s_1 + (1-\theta)s_2$, with $0 \le \theta \le 1$. The result follows by applying the Hölder inequality with $p = 1/\theta$ and $q = 1/(1-\theta)$. Indeed,

$$\begin{aligned}\|f\|_{s,2}^2 &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{\theta s_1+(1-\theta)s_2} |\widehat{f}(\xi)|^{2\theta} |\widehat{f}(\xi)|^{2(1-\theta)}\, d\xi \\ &\le \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s_1} |\widehat{f}(\xi)|^2\, d\xi \right)^{\theta} \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^{s_2} |\widehat{f}(\xi)|^2\, d\xi \right)^{(1-\theta)} \\ &= \|f\|_{s_1,2}^{2\theta} \|f\|_{s_2,2}^{2(1-\theta)}.\end{aligned}$$

□

Proposition 2. *The topological dual of* $H^s(\mathbb{R}^n)$*, denoted by* $(H^s(\mathbb{R}^n))'$*, is isometrically isomorphic to* $H^{-s}(\mathbb{R}^n)$ *by the map*

$$\begin{aligned} \alpha : H^{-s}(\mathbb{R}^n) &\to (H^s(\mathbb{R}^n))' \\ f &\mapsto : H^s(\mathbb{R}^n) \to \mathbb{C} \\ & \qquad g \mapsto \langle f, g\rangle_{-s,s} = \int_{\mathbb{R}^n} \widehat{f}(\xi)\,\widehat{g}(\xi)\,d\xi. \end{aligned}$$

$\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. Given $g \in H^s(\mathbb{R}^n)$, let g_n be defined by

$$\widehat{g}_n(\xi) = \begin{cases} \widehat{g}(\xi), & \text{if } |\xi| \le n \\ 0, & \text{otherwise.} \end{cases}$$

Then $g_n \in \mathcal{S}(\mathbb{R}^n)$ and

$$\begin{aligned} \|g - g_n\|_{s,2}^2 &= \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{g}(\xi) - \widehat{g}_n(\xi)|^2 \\ &= \int_{|\xi|>n} (1 + |\xi|^2)^s |\widehat{g}(\xi)|^2 \, d\xi \to 0 \end{aligned}$$

as $n \to \infty$ since $g \in H^s(\mathbb{R}^n)$.

To understand the relationship between the spaces $H^s(\mathbb{R}^n)$ and the differentiability of functions in $L^2(\mathbb{R}^n)$, we recall the definition of L^p derivative in the case $p = 2$.

Definition 2. *A function f is differentiable in $L^2(\mathbb{R}^n)$ with respect to the k-th variable if there exists $g \in L^2(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \left| \frac{f(x + h\, e_k) - f(x)}{h} - g(x) \right|^2 dx \to 0 \quad \textit{when} \quad h \to 0,$$

where e_k has k-th coordinate equal to 1 and zero in the others. Equivalently (Exercise) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, or

$$\int_{\mathbb{R}^n} f(x) \partial_{x_k} \phi(x)\, dx = - \int_{\mathbb{R}^n} g(x) \phi(x)\, dx$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$ ($C_0^\infty(\mathbb{R}^n)$ being the space of functions infinitely differentiable with compact support).

Example 5. *Let* $n = 1$ *and* $f(x) = \chi_{(-1,1)}(x)$, *then* $f' = \delta_{-1} - \delta_1$, *where* δ_x *represents the measure of mass* 1 *concentrated in* x, *therefore* $f' \notin L^2(\mathbb{R})$.

Example 6. *Let* $n = 1$ *and* g *be as in Example 2. Then*

$$\frac{dg}{dx}(x) = \chi_{(-2,0)} - \chi_{(0,2)}, \quad \text{and so} \quad \frac{dg}{dx} \in L^2(\mathbb{R}).$$

With this definition, for $k \in \mathbb{Z}^+$ we can give a description of the space $H^k(\mathbb{R}^n)$ without using the Fourier transform.

Theorem 1. *If* k *is a positive integer, then* $H^k(\mathbb{R}^n)$ *coincides with the space of functions* $f \in L^2(\mathbb{R}^n)$ *whose derivatives (in the distribution sense)* $\partial_x^\alpha f$ *belong to* $L^2(\mathbb{R}^n)$ *for every* $\alpha \in (\mathbb{Z}^+)^n$ *with* $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$.

In this case the norms $\|f\|_{k,2}$ *and* $\Big(\sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_2 \Big)^{1/2}$ *are equivalent.*

Proof. The proof follows by combining the formula

$$\widehat{\partial_x^\alpha f}(\xi) = (2\pi i\xi)^\alpha \widehat{f}(\xi) \tag{0.2}$$

and the inequalities

$$|\xi^\beta| \ \le (1+|\xi|^2)^{k/2} \ \le \sum_{|\alpha|\le k} |\xi^\alpha|, \quad \beta \in (\mathbb{Z}^+)^n, \ \ |\beta| \le k. \tag{0.3}$$

In fact, let $f \in H^k(\mathbb{R}^n)$, then using (0.3) we obtain

$$|(i\xi)^\beta \widehat{f}(\xi)| = |\xi^\beta||\widehat{f}(\xi)| \le (1+|\xi|^2)^{k/2}|\widehat{f}(\xi)| \quad \beta \in (\mathbb{Z}^+)^n, \ \ |\beta| \le k,$$

which implies that $\partial_x^\beta f \in L^2(\mathbb{R}^n)$ for any $\beta \in (\mathbb{Z}^+)^n$ with $|\beta| \le k$. Thus

$$\sum_{|\alpha|\le k} \|\partial_x^\alpha f\|_2 \le c_k \|f\|_{k,2}^2.$$

□

If $\partial_x^\alpha f \in L^2(\mathbb{R}^n)$ for any $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| \leq k$ we have from (0.2) that $(2\pi i\xi)^\alpha \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$ for any $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| \leq k$. Then

$$\begin{aligned}
\|f\|_{k,2}^2 &= \int_{\mathbb{R}^n} (1+|\xi|^2)^k \, |\widehat{f}(\xi)|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^n} \Big(\sum_{|\alpha|\leq k} |\xi^\alpha|\Big)^2 \, |\widehat{f}(\xi)|^2 \\
&\leq \sum_{|\alpha|\leq k} c_k \int_{\mathbb{R}^n} \Big(|(i\xi)^\alpha \widehat{f}(\xi)|^2 \\
&\leq C \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_2.
\end{aligned}$$

Theorem 1 allows us to define in a natural manner $H^k(\Omega)$, the Sobolev space of order $k \in \mathbb{Z}^+$ in any subset Ω (open) of $\mathbb{R}^n$. Given $f \in L^2(\Omega)$ we say that $\partial_x^\alpha f$, $\alpha \in (\mathbb{Z}^+)^n$ is the αth-partial derivative (in the distribution sense) of f if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega f \partial_x^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega \partial_x^\alpha f \, \phi \, dx.$$

Then

$$H^k(\Omega) = \{f \in L^2(\Omega) \ : \ \partial_x^\alpha f \text{ (in the distribution sense) } \in L^2(\Omega), \ |\alpha| \leq k\}$$

with the norm

$$\|f\|_{H^k(\Omega)} \equiv \Big(\sum_{|\alpha| \leq k} \int_\Omega |\partial_x^\alpha f(x)|^2 \, dx \Big)^{1/2}.$$

Example 7. *For* $n = 1$, $b > 0$ *and* $f(x) = |x|$ *one has that* $f \in H^1((-b, b))$ *and* $f \notin H^2((-b, b))$.

The next result allows us to relate “weak derivatives" with derivatives in the classical sense.

Theorem 2 (Embedding)**.** *If* $s > n/2 + k$, *then* $H^s(\mathbb{R}^n)$ *is continuously embedded in* $C^k_\infty(\mathbb{R}^n)$*, the space of functions with* k *continuous derivatives vanishing at infinity. In other words, if* $f \in H^s(\mathbb{R}^n)$*,* $s > n/2 + k$*, then (after a possible modification of* f *in a set of measure zero)* $f \in C^k_\infty(\mathbb{R}^n)$ *and*

$$\|f\|_{C^k} \leq c_s \|f\|_{s,2}. \tag{0.4}$$

Proof. Case $k = 0$: we first show that if $f \in H^s(\mathbb{R}^n)$ then $\widehat{f} \in L^1(\mathbb{R}^n)$ with

$$\|\widehat{f}\,\|_1 \leq c_s \|f\|_{s,2} \quad \text{if} \quad s > n/2. \tag{0.5}$$

Using the Cauchy–Schwarz inequality we deduce

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)| d\xi = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|(1 + |\xi|^2)^{s/2} \frac{d\xi}{(1 + |\xi|^2)^{s/2}}$$

$$\leq \|\Lambda^s f\|_2 \left(\int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|^2)^s} \right)^{1/2} \leq c_s \|f\|_{s,2}$$

□

if $s > n/2$. Combining (0.5), Proposition 1.2, and Theorem 1.1 we conclude that

$$\|f\|_\infty = \|(\widehat{f}\,)^\vee\|_\infty \le \|\widehat{f}\,\|_1 \le c_s \|f\|_{s,2}.$$

Case $k \ge 1$: Using the same argument we have that if $f \in H^s(\mathbb{R}^n)$ with $s > n/2 + k$, then for $\alpha \in (\mathbb{Z}^+)^n$, $|\alpha| \le k$, it follows that $\widehat{\partial_x^\alpha f} \in L^1(\mathbb{R}^n)$ and

$$\|\partial_x^\alpha f\|_\infty \le \|\widehat{\partial_x^\alpha f}\|_1 = \|(2\pi i\xi)^\alpha \widehat{f}\,\|_1 \le c_s \|f\|_{s,2}.$$

$\square$

Corollary 1. *If* $s = n/2 + k + \theta$, *with* $\theta \in (0, 1)$, *then* $H^s(\mathbb{R}^n)$ *is continuously embedded in* $C^{k+\theta}(\mathbb{R}^n)$, *the space of* C^k *functions with partial derivatives of order* k *Hölder continuous with index* θ.

Proof. We only prove the case $k = 0$ since the proof of the general case follows the same argument. From the formula of inversion of the Fourier transform and the Cauchy–Schwarz inequality we have

$$|f(x+y) - f(x)| = \Big| \int_{\mathbb{R}^n} e^{2\pi i (x\cdot\xi)} \widehat{f}(\xi)(e^{2\pi i (y\cdot\xi)} - 1)\, d\xi \Big|$$

$$\leq \Big(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{n/2+\theta} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \frac{|e^{2\pi i (y\cdot\xi)} - 1|^2}{(1 + |\xi|^2)^{n/2+\theta}} \, d\xi \Big)^{1/2}.$$

□

But

$$\int_{\mathbb{R}^n} \frac{|e^{2\pi i(y\cdot\xi)} - 1|^2}{(1+|\xi|^2)^{n/2+\theta}}\, d\xi$$

$$\leq c \int_{|\xi|\leq|y|^{-1}} |y|^2|\xi|^2 \frac{d\xi}{(1+|\xi|^2)^{n/2+\theta}} + 4\int_{|\xi|\geq|y|^{-1}} \frac{d\xi}{(1+|\xi|^2)^{n/2+\theta}}$$

$$\leq c|y|^2 \int_0^{|y|^{-1}} \frac{r^{n+1}}{(1+r)^{n+2\theta}}\, dr + 4\int_{|y|^{-1}}^{\infty} \frac{r^{n-1}}{(1+r)^{n+2\theta}}\, dr \;\leq\; c\,|y|^{2\theta}.$$

If $|y| < 1$ we conclude that $|f(x+y) - f(x)| \leq c\,|y|^{\theta}$. This finishes the proof. □

Theorem 3. *If* $s \in (0, n/2)$, *then* $H^s(\mathbb{R}^n)$ *is continuously embedded in* $L^p(\mathbb{R}^n)$ *with* $p = 2n/(n-2s)$, *i.e.,* $s = n(1/2 - 1/p)$. *Moreover, for* $f \in H^s(\mathbb{R}^n)$, $s \in (0, n/2)$,

$$\|f\|_p \le c_{n,s} \|D^s f\|_2 \le c\|f\|_{s,2}, \tag{0.6}$$

where

$$D^l f = (-\Delta)^{l/2} f = ((2\pi|\xi|)^l \widehat{f}\,)^\vee.$$

Proof. The last inequality in (0.6) is immediate so we just need to show the first one. We define

$$D^s f = g \quad \text{or} \quad f = D^{-s} g = c_{n,s} \left(\frac{1}{|\xi|^s} \widehat{g} \right)^{\vee} = \frac{c_{n,s}}{|x|^{n-s}} * g, \tag{0.7}$$

where we have used the result of Exercise 1.14. Thus by the Hardy–Littlewood–Sobolev estimate (0.20) it follows that

$$\|f\|_p = \|D^{-s} g\|_p = \left\| \frac{c_{n,s}}{|x|^{n-s}} * g \right\|_p \leq c_{n,s} \|g\|_2 = c\|D^s f\|_2. \tag{0.8}$$

□

We notice from Theorems 2 and 3, and Corollary 1 that local regularity in H^s, $s > 0$, increases with the parameter s.

Examples 1 and 3 show that the functions in $H^s(\mathbb{R}^n)$ with $s < n/2$ or $s < n/2 + 1$ respectively are not necessarily continuous nor C^1. Moreover, let $f \in L^2(\mathbb{R}^n)$ with

$$\widehat{f}(\xi) = \frac{1}{(1 + |\xi|)^n \, \log(2 + |\xi|)}$$

(which is radial, decreasing and positive). A simple computation shows that $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$, but $\widehat{f} \notin L^1(\mathbb{R}^n)$ and so $f \notin L^\infty(\mathbb{R}^n)$ since $f(0) = \int \widehat{f}(\xi) d\xi = \infty$ (see also Exercise 3.11 (iii)).

To complete the embedding results of the spaces $H^s(\mathbb{R}^n)$, $s > 0$, it remains to consider the case $s = n/2$ (since for $s = k + n/2$, $k \in \mathbb{Z}^+$, the result follows from this one). So we define the space of functions of bounded mean oscillation or BMO.

Definition 3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we say that $f \in \mathbf{BMO}(\mathbb{R}^n)$ (f has bounded mean oscillation) if*

$$\|f\|_{\mathrm{BMO}} = \sup_{\substack{x\in\mathbb{R}^n \\ r>0}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|\, dy < \infty \tag{0.9}$$

where

$$f_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy.$$

Notice that $\|\cdot\|_{\mathrm{BMO}}$ is a semi-norm since it vanishes for constant functions.

$\mathbf{BMO}(\mathbb{R}^n)$ is a vector space with $L^\infty(\mathbb{R}^n) \subsetneq \mathbf{BMO}(\mathbb{R}^n)$, since $\|f\|_{\mathbf{BMO}} \leq 2\|f\|_\infty$ and $\log|x| \in \mathbf{BMO}(\mathbb{R}^n)$.

Theorem 4. *$H^{n/2}(\mathbb{R}^n)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^n)$. More precisely, there exists $c = c(n) > 0$ such that*

$$\|f\|_{\mathrm{BMO}} \leq c\,\|D^{n/2}f\|_2.$$

Proof. Without loss of generality we assume f real valued. Consider $x \in \mathbb{R}^n$ and $r > 0$.
Let $\phi_r \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi_r \subseteq \{x \mid |x| \leq \frac{2}{r}\}$ with $0 \leq \phi_r(x) \leq 1$ and $\phi_r(x) \equiv 1$ if $|x| < 1/r$, and define

$$f(x) = f_l + f_h = \left(\widehat{f}\phi_r\right)^\vee + \left(\widehat{f}(1-\phi_r)\right)^\vee.$$

We observe that

$$\|f\|_{\mathrm{BMO}} \leq \|f_l\|_{\mathrm{BMO}} + \|f_h\|_{\mathrm{BMO}}$$

□

and $f_l \in H^s(\mathbb{R}^n)$ for any $s > 0$, therefore

$$f_{l,B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f_l(y)\, dy = f_l(x_0)$$

for some $x_0 \in B_r(x)$, and so for any $y \in B_r(x)$

$$|f_l(y) - f_{l,B_r(x)}| \leq 2r \, \|\nabla f_l\|_\infty.$$

□

Using this estimate we get

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} |f_l(y) - f_{l,B_r(x)}|\, dy \\ &\le \frac{1}{|B_r(x)|^{1/2}}\Big(\int_{B_r(x)} |f_l(y) - f_{l,B_r(x)}|^2\, dy\Big)^{1/2} \\ &\le 2r\,\|\nabla f_l\|_\infty \le 2r\,\|\widehat{\nabla f_l}\|_1 \\ &\le 2r \int_{|\xi|\le 1/2r} |\xi|^{1-n/2}|\xi|^{n/2}|\widehat{f}(\xi)|\, d\xi \\ &\le 2r\Big(\int_{|\xi|\le 1/2r} |\xi|^{2-n}\, d\xi\Big)^{1/2}\|D^{n/2}f\|_2 \le c\|D^{n/2}f\|_2.\end{aligned}$$

Also

$$
\begin{aligned}
\frac{1}{|B_r(x)|} \int_{B_r(x)} |f_h(y) - f_{h,B_r(x)}| \, dy
&\le \frac{2}{|B_r(x)|^{1/2}} \|f_h\|_2 \\
&\le \frac{2}{|B_r(x)|^{1/2}} \left(\int_{|\xi| \ge 1/2r} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} \\
&= \frac{c_n}{r^{n/2}} \left(\int_{|\xi| \ge 1/2r} r^n |\xi|^n |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} \le \|D^{n/2} f\|_2,
\end{aligned}
$$

which yields the desired result. □

We have shown that $H^s(\mathbb{R}^n)$ with $s > n/2$ is a Hilbert space whose elements are continuous functions. From the point of view of nonlinear analysis the next property is essential.

Theorem 5. *If* $s > n/2,$ *then* $H^s(\mathbb{R}^n)$ *is an algebra with respect to the product of functions. That is, if* $f, g \in H^s(\mathbb{R}^n),$ *then* $fg \in H^s(\mathbb{R}^n)$ *with*

$$\|fg\|_{s,2} \leq c_s \|f\|_{s,2} \|g\|_{s,2}. \tag{0.10}$$

Proof. From the triangle inequality we have that for every $\xi, \eta \in \mathbb{R}^n$

$$(1+|\xi|^2)^{s/2} \leq 2^s[(1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2}]. \tag{0.11}$$

Using this we deduce that

$$\begin{aligned}
|\Lambda^s(fg)| &= |(1+|\xi|^2)^{s/2}\widehat{(fg)}(\xi)| \\
&= (1+|\xi|^2)^{s/2}\,|\int_{\mathbb{R}^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta| \\
&\leq 2^s \int_{\mathbb{R}^n} [(1+|\xi-\eta|^2)^{s/2}\,|\widehat{f}(\xi-\eta)\widehat{g}(\eta)| \\
&\quad + (1+|\eta|^2)^{s/2}\,|\widehat{f}(\xi-\eta)\widehat{g}(\eta)|]\,d\eta \\
&\leq 2^s(|\widehat{\Lambda^s f}| * |\widehat{g}| + |\widehat{f}| * |\widehat{\Lambda^s g}|).
\end{aligned}$$

□

Thus, taking the L^2-norm and using Young's inequality it follows that

$$\|fg\|_{s,2} = \|\Lambda^s(fg)\|_2 \leq c(\|\Lambda^s f\|_2 \| \widehat{g} \|_1 + \| \widehat{f} \|_1 \|\Lambda^s g\|_2). \tag{0.12}$$

Finally, (0.5) assures one that if $r > n/2$, then

$$\begin{aligned} \|fg\|_{s,2} &\leq c_s(\|f\|_{s,2}\| \widehat{g} \|_1 + \| \widehat{f} \|_1 \|g\|_{s,2}) \\ &\leq c_s(\|f\|_{s,2}\|g\|_{r,2} + \|f\|_{r,2}\|g\|_{s,2}). \end{aligned} \tag{0.13}$$

Choosing $r = s$ we obtain (0.10). □

The inequality (0.13) is not sharp as the following scaling argument shows. Let $\lambda > 0$ and

$$f(x) = f_1(\lambda x), \quad g(x) = g_1(\lambda x), \quad f_1,\, g_1 \in \mathcal{S}(\mathbb{R}^n).$$

Then as $\lambda \uparrow \infty$ the right hand side of (0.13) grows as λ^{s+r}, meanwhile the left hand side grows as λ^s. This will not be the case if we replace $\|\cdot\|_{r,2}$ in (0.13) with the $\|\cdot\|_\infty$-norm to get that

$$\|fg\|_{s,2} \leq c_s(\|f\|_{s,2}\,\|g\|_\infty + \|f\|_\infty\,\|g\|_{s,2}) \tag{0.14}$$

which in particular shows that for any $s > 0$, $H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is an algebra under the pointwise product.

For $s \in \mathbb{Z}^+$, the inequality (0.14) follows by combining the Leibniz rule for the product of functions and the Gagliardo–Nirenberg inequality:

$$\|\partial_x^\alpha f\|_p \;\le\; c \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta \; \|f\|_r^{1-\theta} \tag{0.15}$$

with $|\alpha| = j$, $c = c(j, m, p, q, r)$, $1/p - j/n = \theta(1/q - m/n) + (1 - \theta)1/r$, $\theta \in [j/m, 1]$. For the proof of this inequality we refer the reader to the reference [3].

Easy example, take $f \in C_0^1(\mathbb{R})$,

$$f^2(x) = \int_a^x \frac{d}{dy} f^2(y)\, dy = 2\int_a^x f(y) \frac{d}{dy} f(y)\, dy.$$

Using Cauchy-Schwarz inequality we find that

$$|f(x)|^2 \leq 2\|f\|_{L^2}\|f'\|_{L^2}.$$

Thus

$$\|f\|_{L^\infty} \leq \sqrt{2}\|f\|_{L^2}^{\frac{1}{2}}\|f'\|_{L^2}^{\frac{1}{2}}.$$

For the general case $s > 0$ where the usual pointwise Leibniz rule is not available, the inequality (0.14) still holds (see [5]). The inequality (0.14) has several extensions, for instance: Let $s \in (0,1)$, $r \in [1,\infty)$, $1 < p_j, q_j \leq \infty$, $1/r = 1/p_j + 1/q_j$, $j = 1,2$. Then

$$\|\Phi^s(fg)\|_r \leq c\big(\|\Phi^s(f)\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|\Phi^s(g)\|_{q_2}\big),$$

with $\Phi^s = \Lambda^s$ or D^s, (for the proof of this estimate and further generalizations [7], [9] and [4]). The extension to the case $r = p_j = q_j = \infty$, $j = 1,2$ was given in [2].

Proposition 3. *If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}$, then the map $u \mapsto \varphi u$ is a bounded linear map of $H^s(\mathbb{R}^n)$ to itself. Moreover,*

$$\|\varphi u\|_{s,2} \leq c_{s,n} \|(1+|\cdot|^2)^{|s|}\widehat{\varphi}(\cdot)\|_{L^1}\|u\|_{s,2}. \tag{0.16}$$

Proof. We will use the following elementary inequality

$$(1+|\xi|^2)^{\sigma/2} \leq 2^{|\sigma|/2}(1+|\xi-\eta|^2)^{s/2}(1+|\eta|^2)^{s/2}. \tag{0.17}$$

Let $s \geq 0$. Fourier transform properties and inequality (0.17) yield

$$\begin{aligned}(1+|\xi|^2)^{s/2}\widehat{\varphi u}(\xi) &= \int_{\mathbb{R}^n}(1+|\xi|^2)^{s/2}\widehat{\varphi}(\xi-\eta)\widehat{u}(\eta)\,d\eta \\ &\leq \int_{\mathbb{R}^n}\frac{(1+|\xi|^2)^{s/2}}{(1+|\eta|^2)^{s/2}}\widehat{\varphi}(\xi-\eta)(1+|\eta|^2)^{s/2}\widehat{u}(\eta)\,d\eta \\ &\leq c\int_{\mathbb{R}^n}(1+|\xi-\eta|^2)^{s/2}\widehat{\varphi}(\xi-\eta)(1+|\eta|^2)^{s/2}\widehat{u}(\eta)\,d\eta\end{aligned}$$

Young's inequality yields the result. □

For the case $s < 0$, we observe that

$$\frac{(1+|\xi|^2)^{s/2}}{(1+|\eta|^2)^{s/2}} = \frac{(1+|\eta|^2)^{|s|/2}}{(1+|\xi|^2)^{|s|/2}} \leq c(1+|\xi-\eta|^2)^{|s|/2}$$

by employing inequality (0.17).
Then

$$|(1+|\xi|^2)^{s/2}\widehat{\varphi u}(\xi)| \leq c\int_{\mathbb{R}^n} |(1+|\xi-\eta|^2)^{s/2}\widehat{\varphi}(\xi-\eta)||(1+|\eta|^2)^{s/2}\widehat{u}(\eta)|\,d\eta$$

Young's inequality implies

$$\|\varphi u\|_{s,2} \leq c_{s,n}\|(1+|\cdot|^2)^{|s|}\widehat{\varphi}(\cdot)\|_{L^1}\|u\|_{s,2}.$$

In many applications the following commutator estimate is often used

$$\sum_{|\alpha|=s} \|[\partial_x^\alpha; g]\, f\|_2 \equiv \sum_{|\alpha|=s} \|\partial_x^\alpha(gf) - g\partial_x^\alpha\, f\|_2$$

$$\leq c_{n,s}\,\Big(\,\|\nabla g\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta\, f\|_2 + \|f\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta\, g\|_2\,\Big)$$

(see [6]). Similarly, for $s \geq 1$ one has

$$\|[\Lambda^s; g]\, f\|_2 \leq c\,(\|\nabla g\|_\infty \|\Lambda^{s-1} f\|_2 + \|f\|_\infty\, \|\Lambda^s g\|_2),$$

(see [5]).

Here $[\partial_x^\alpha; g]\, f = \partial_x^\alpha(gf) - g\partial_x^\alpha f$.

In general, for two linear operators T, S the commutator of T and S is defined by $[T, S] = TS - ST$.

There are “equivalent" manners to define fractional derivatives without relying on the Fourier transform. For instance:

Definition 4 (Stein [10]). *For* $b \in (0,1)$ *and an appropriate* f *define*

$$\mathcal{D}^b f(x) = \Big(\int \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}}\, dy \Big)^{1/2}. \qquad (0.18)$$

Theorem 6 (Stein [10]). *Let* $b \in (0,1)$ *and* $\frac{2n}{(n+2b)} \le p < \infty$. *Then* $f,\ D^b f \in L^p(\mathbb{R}^n)$ *if and only if* $f,\ \mathcal{D}^b f \in L^p(\mathbb{R}^n)$.
Moreover,

$$\|f\|_p + \|D^b f\|_p \sim \|f\|_p + \|\mathcal{D}^b f\|_p.$$

The case $p = 2$ was previously considered in [1].
For other “equivalent" definitions of fractional derivatives see [11].

Finally, to complete our study of Sobolev spaces we introduce the localized Sobolev spaces.

Definition 5. *Given* $f : \mathbb{R}^n \to \mathbb{R}$ *we say that* $f \in H^s_{loc}(\mathbb{R}^n)$ *if for every* $\varphi \in C_0^\infty(\mathbb{R}^n)$ *we have* $\varphi f \in H^s(\mathbb{R}^n)$. *In other words, for any* $\Omega \subseteq \mathbb{R}^n$ *open bounded* $f|_\Omega$ *coincides with an element of* $H^s(\mathbb{R}^n)$.

This means that f has the sufficient regularity but may not have enough decay to be in $H^s(\mathbb{R}^n)$.

Example 8. *Let* $n = 1, \quad f(x) = x,$ *and* $g(x) = |x|$, *then* $f \in H^s_{\text{loc}}(\mathbb{R})$ *for every* $s \geq 0$ *and* $g \in H^s_{\text{loc}}(\mathbb{R})$ *for every* $s < 3/2$.

Hardy-Littlewood-Sobolev Theorem

Definition 6. Let $0 < \alpha < n$. *The Riesz potential of order* α, denoted by I_α, is defined as

$$I_\alpha f(x) = c_{\alpha,n} \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy = k_\alpha * f(x), \qquad (0.19)$$

where $c_{\alpha,n} = \pi^{-n/2} 2^{-\alpha}\, \Gamma(n/2 - \alpha/2)/\Gamma(\alpha/2)$.

Since the Riesz potentials are defined as integral operators it is natural to study their continuity properties in $L^p(\mathbb{R}^n)$.

Theorem 7 (Hardy–Littlewood–Sobolev). *Let* $0 < \alpha < n$, $1 \leq p < q < \infty$, *with* $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{\alpha}{n}$.

1. *If* $f \in L^p(\mathbb{R}^n)$, *then the integral (0.19) is absolutely convergent almost every* $x \in \mathbb{R}^n$.

2. *If* $p > 1$, *then* I_α *is of type (p,q), i.e.,*

$$\|I_\alpha(f)\|_q \leq c_{p,\alpha,n}\|f\|_p. \tag{0.20}$$

For a proof of this theorem see [8].

References

[1] N. Aronszajn and K. T. Smith. *Theory of Bessel potentials I*, Ann. Inst. Fourier (Grenoble) **11** (1961), 385–475.

[2] J. Bourgain and D. Li. *On an endpoint Kato-Ponce inequality*, Diff. and Int. Eqs. **27** (2014), no. 11-12, 1037–1072.

[3] A. Friedman. *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1976.

[4] L. Grafakos and S. Oh. *Kato-Ponce inequality*, Comm. PDE. **39** (2014), no. 6, 1128–1157.

[5] T. Kato and G. Ponce. *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), 891–907.

[6] S. Klainerman. *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math. **33** (1980), 43–101.

[7] C.E. Kenig, G. Ponce, and L. Vega. *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[8] F. Linares and G. Ponce. *Introduction to Nonlinear Dispersive Equations*, Second edition. Universitext. Springer, New York, 2015. xiv+301.

[9] C. Muscalu, J.Pipher, T. Tao, and C. Thiele, *Bi-parameter paraproducts*. Acta Math. **193** (2004), no. 2, 269–296.

[10] E.M. Stein. *The characterization of functions arising as potentials*, Bull. A.M.S. **67**, (1961), 102–104.

[11] R.S. Strichartz. *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), 1031–1060.