The Hille-Yosida Theorem

Semigroup theory is the abstract study of first-order ordinary differential equations with values in Banach spaces, driven by linear, but possibly unbounded operators.

Definition 1. *Let X be a Banach space. A one parameter family of $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X to X is a strongly continuous semigroup of bounded linear operators on X if*

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$,

(iii) $\lim_{t \downarrow 0} T(t)x = x$ *for every* $x \in X$.

We will use C_0 semigroup to connote a strongly continuous semigroup.

Definition 2. *A linear operator A defined by*

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\} \tag{0.1}$$

and

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A) \tag{0.2}$$

is called the infinitesimal generator of the semigroup $T(t)$.

Theorem 1. *Let $T(t)$ be a C_0 semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq M\, e^{\omega t} \quad \text{for } 0 \leq t < \infty. \tag{0.3}$$

Proof. First we prove that there exists $\eta > 0$ such that $\|T(t)\|$ is bounded for $0 \leq t \leq \eta$. By contradiction, we suppose this is false. Then there is a sequence $\{t_n\}$ satisfying $t_n \geq 0$, $\lim_{n\to\infty} t_n = 0$ and $\|T(t_n)\| \geq n$. From uniform boundedness theorem it then follows that for some $x \in X$, $\|T(t_n)x\|$ is unbounded contrary to (iii) in Definition 1. Hence $\|T(t)\| \leq M$ for $0 \leq t \leq \eta$. Since $\|T(0)\| = 1$, $M \geq 1$. Let $\omega = \dfrac{\log M}{\eta} \geq 0$. Given $t \geq 0$ we have $t = n\eta + \delta$ where $0 \leq \delta < \eta$ and therefore by the semigroup property

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \leq M^{n+1} \leq M M^{t/\eta} = M e^{\omega t}.$$

□

Corollary 1. *If $T(t)$ is a C_0 semigroup then for every $x \in X$, $t \mapsto T(t)x$ is a continuous function from $[0, \infty)$ into X.*

Proof. Let $t, h \geq 0$. We observe that

$$\|T(t+h)x - T(t)x\| \leq \|T(t)\|\|T(h)x - x\| \leq Me^{\omega t}\|T(h)x - x\|$$

and for $t \geq h \geq 0$ that

$$\|T(t-h)x - T(t)x\| \leq \|T(t-h)\|\|x - T(h)x\| \leq Me^{\omega t}\|x - T(h)x\|.$$

Then the continuity follows. □

Theorem 2. *Let* $T(t)$ *be a* C_0 *semigroup and let* A *be its infinitesimal generator. Then*

(i) For $x \in X$*,*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x\,ds = T(t)x. \tag{0.4}$$

(ii) For $x \in X$*,* $\int_0^t T(s)x\,ds \in D(A)$ *and*

$$A\Big(\int_0^t T(s)x\,ds \Big) = T(t)x - x. \tag{0.5}$$

(iii) For $x \in D(A)$*,* $T(t)x \in D(A)$ *and*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax. \tag{0.6}$$

(iv) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax\,d\tau = \int_s^t AT(\tau)x\,d\tau. \tag{0.7}$$

Proof.
Part (i) follows from the continuity.
To show part (ii) let $x \in X$ and $h > 0$. We write

$$\begin{aligned} \frac{T(h)-I}{h}\int_0^t T(s)x\,ds &= \frac{1}{h}\int_0^t \big(T(s+h)x - T(s)x\big)\,ds \\ &= \frac{1}{h}\int_t^{t+h} T(s)x\,ds - \frac{1}{h}\int_0^h T(s)x\,ds. \end{aligned}$$

Now taking $h \downarrow 0$ the right hand side tends to $T(t)x - x$ whence (ii) follows.

Now we prove (iii). Let $x \in D(A)$ and $h > 0$.Then

$$\frac{T(h)-I}{h}T(t)x = T(t)\frac{T(h)-I}{h}x \to T(t)Ax \quad \text{as} \quad h \downarrow 0. \tag{0.8}$$

Hence, $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$. From (0.8) we also deduce that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax.$$

Now we need to verify that for $t > 0$, the left derivative exists and equals to $T(t)Ax$. To do so, we write

$$\lim_{h\downarrow 0}\Big(\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\Big) = \lim_{h\downarrow 0} T(t-h)\Big(\frac{T(h)x - x}{h} - Ax\Big) + \lim_{h\downarrow 0}\Big(T(t-h)Ax - T(t)Ax\Big)$$

We notice that both terms on the right hand side are zero. The first one due to the fact that $x \in D(A)$ and $\|T(t-h)\|$ is bounded on $0 \le h \le t$. The second one because of the strong continuity of $T(t)$. Thus property (iii) follows.

Part (iv) can be deduced by integrating (0.6) from 0 to t.

Corollary 2. *If A is the infinitesimal generator of a C_0 semigroup $T(t)$ then $D(A)$ is dense in X and A is a closed linear operator.*

Proof. For every $x \in X$ we set $x_t = \frac{1}{t}\int_0^t T(s)x\,ds$.

From (ii) in Theorem 2 $x_t \in D(A)$ for all $t > 0$ and by (i) $x_t \to x$ as $t \downarrow 0$. Therefore $\overline{D(A)} = X$. The operator A is clearly linear. Now we show that it is closed. Let $x_n \in D(A)$, $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. By (iv) in Theorem 2 we have that

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n\,ds. \tag{0.9}$$

The integrand converges to $T(s)y$ uniformly on bounded intervals.

Thus letting $n \to \infty$ in (0.9) we obtain

$$T(t)x - x = \int_0^t T(s)y\,ds. \tag{0.10}$$

Dividing the last identity by $t > 0$ and making $t \downarrow 0$, we deduce by employing (i) in Theorem 2 that $x \in D(A)$ and $Ax = y$.

□

Theorem 3. *Let $T(t)$ and $S(t)$ be C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$ for $t \geq 0$.*

Proof. Let $x \in D(A) = D(B)$. By (iii) in Theorem 2 we deduce that the function $s \to T(t-s)S(s)x$ is differentiable and that

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x \\ &= -T(t-s)AS(s)x + T(t-s)BS(s)x = 0.\end{aligned}$$

Therefore the function $s \to T(t-s)S(s)x$ is constant and in particular its values at $s = 0$ and $s = t$ are the same, that is, $T(t)x = S(t)x$. This is true for any $x \in D(A)$ since from Corollary 2 $D(A)$ is dense in X and $T(t)$ and $S(t)$ are bounded, $T(t)x = S(t)x$ for every $x \in X$. □

If A is the infinitesimal generator of a C_0 semigroup then by Corollary 2, $\overline{D(A)} = X$. A stronger result can be proved.

Theorem 4. *Let A be the infinitesimal generator of the C_0 semigroup $T(t)$. If $D(A^n)$ is the domain de A^n, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.*

Proof. See Theorem 2.7 in [3]. □

Exercise 5. *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leq M$ for $t \geq 0$. If $x \in D(A^2)$, show that*

$$\|Ax\|^2 \leq 4M^2\|A^2x\|\|x\|. \tag{0.11}$$

Definition 3. *A* C_0 *semigroup of operators* $T(t)$ *is called a* C_0 *semigroup of contractions if*

$$\|T(t)\| \leq 1.$$

i.e. this corresponds to $M = 1$ *and* $\omega = 0$ *in* (0.3).

In the next the characterization of the infinitesimal generators of C_0 semigroups of contractions will be established.

Theorem 6 (Hille-Yosida). *A linear (unbounded) operator* A *is the infinitesimal generator of a* C_0 *semigroup of contractions* $T(t)$, $t \geq 0$ *if and only if*

(i) A *is closed and* $\overline{D(A)} = X$.

(ii) $(0, \infty) \subset \rho(A)$ *and*

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda}. \tag{0.12}$$

Proof of Hille-Yosida Theorem

Necessity. By Corollary 2 we have A closed and $\overline{D(A)} = X$. For $\lambda > 0$ and $x \in X$, set

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T(t) x \, dt. \tag{0.13}$$

Since $t \to T(t)x$ is continuous and uniformly bounded the integral exists as an improper Riemann integral and defines a bounded linear operator R_λ with

$$\|R_\lambda x\| = \int_0^\infty e^{-\lambda t} \|T(t)x\| \, dt \leq \frac{1}{\lambda}\|x\|. \tag{0.14}$$

In addition, for $h > 0$

$$\begin{aligned}\frac{T(h)-I}{h} R_\lambda x &= \frac{1}{h}\int_0^\infty e^{-\lambda t}\left(T(t+h)x - T(t)x\right) dt \\ &= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}\, T(t)x\, dt - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda t}\, T(t)x\, dt.\end{aligned} \tag{0.15}$$

as $h \downarrow 0$, the right hand side of (0.15) converges to $\lambda R_\lambda x - x$. This implies that for every $x \in X$ and $\lambda > 0$, $R_\lambda x \in D(A)$ and $AR_\lambda = \lambda R_\lambda - I$ or

$$(\lambda I - A)\, R_\lambda = I. \tag{0.16}$$

For $x \in D(A)$ we have

$$\begin{aligned} R_\lambda Ax &= \int_0^\infty e^{-\lambda t} T(t) Ax\, dt = \int_0^\infty e^{-\lambda t} AT(t)x\, dt \\ &= A\Big(\int_0^\infty e^{-\lambda t} T(t)x\, dt\Big) = AR_\lambda x, \end{aligned} \tag{0.17}$$

where we have used (iii) in Theorem 2 and the fact that A is closed. From (0.16) and (0.17) we conclude that

$$R_\lambda(\lambda I - A)x = x \quad \text{for } x \in D(A).$$

Thus, R_λ is the inverse of $\lambda I - A$, it exists for all $\lambda > 0$ and satisfies the estimate (0.12). Therefore conditions (i) and (ii) are necessary.

To prove the sufficiency we need some preparation.

Lemma 1. *Let A satisfy the conditions (i) and (ii) and let $R_\lambda(A) = (\lambda I - A)^{-1}$. Then*

$$\lim_{\lambda\to\infty} \lambda\, R_\lambda(A)x = x \quad \textit{for } x \in X. \tag{0.18}$$

Proof. Since $\lambda R_\lambda(A)x - x = AR_\lambda(A)x = R_\lambda(A)Ax$,

$$\|\lambda R_\lambda(A)x - x\| \le \|R_\lambda(A)\|\|Ax\| \le \frac{1}{\lambda}\|Ax\| \to 0 \ \text{ as } \ \lambda \to \infty.$$

Thus $R_\lambda(A)x \to x$ as $\lambda \to \infty$ if $x \in D(A)$. But since $\|\lambda R_\lambda(A)\| \le 1$ and $D(A)$ is dense, we deduce then as well

$$\lambda R_\lambda(A)x \to x \quad \text{as} \quad \lambda \to \infty \quad \text{for all} \quad x \in X.$$

□

We define, for $\lambda > 0$, the Yosida approximation of A by

$$A_\lambda = \lambda A R_\lambda(A) = \lambda^2 R_\lambda(A) - \lambda I. \tag{0.19}$$

Lemma 2. *Let A satisfy the conditions (i) and (ii). If A_λ is the Yosida approximation of A. Then*

$$\lim_{\lambda\to\infty} A_\lambda x = Ax \quad \textit{for } x \in D(A). \tag{0.20}$$

Proof. If $x \in D(A)$ we observe that

$$A_\lambda x = \lambda A R_\lambda(A) = \lambda R_\lambda(A) A x.$$

Then using (0.20) the result follows. □

Exercise 7. *A semigroup of bounded linear operator, $T(t)$, is called* *uniformly continuous* *if*

$$\lim_{t \downarrow 0} \|T(t) - I\| = 0.$$

Prove that a linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Lemma 3. *Let A satisfy the conditions (i) and (ii). If A_λ is the Yosida approximation of A, then A_λ is the infinitesimal generator of a uniformly continuous semigroup of contractions e^{tA_λ}. In addition, for every $x \in X$, λ, $\mu > 0$ we have*

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \leq t\|A_\lambda x - A_\mu x\|. \tag{0.21}$$

Proof. We notice that A_λ in (0.19) is a bounded linear operator and by Exercise 7 it is the infinitesimal generator of a uniformly continuous semigroup e^{tA_λ} of bounded linear operators. On the other hand,

$$\|e^{tA_\lambda}\| = e^{-t\lambda}\|e^{t\lambda^2 R_\lambda(A)}\| \leq e^{-t\lambda}e^{t\lambda^2\|R_\lambda(A)\|} \leq 1$$

which tells us that e^{tA_λ} is a semigroup of contractions.

We also observe that e^{tA_λ}, e^{tA_μ}, A_λ and A_μ commute with each other. Thus

$$\begin{aligned}\|e^{tA_\lambda}x - e^{tA_\mu}x\| &= \Big\| \int_0^1 \frac{d}{ds}(e^{tsA_\lambda}e^{t(1-s)A_\mu}x)\,ds \Big\| \\ &\le t\| \int_0^1 e^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\,ds\| \\ &\le t\|(A_\lambda x - A_\mu x)\|.\end{aligned}$$

Sufficiency. Let $x \in D(A)$. Then

$$\|e^{tA_\lambda}x - e^{tA_\mu}x\| \le t\|A_\lambda x - A_\mu x\| \le t\|A_\lambda x - Ax\| + t\|Ax - A_\mu x\|.$$

Which implies by Lemma 2 for $x \in D(A)$, that $e^{tA_\lambda}x$ converges uniformly as $\lambda \to \infty$ in bounded intervals. Since $D(A)$ is dense in X and $\|e^{tA_\lambda}\| \le 1$ we deduce

$$\lim_{\lambda\to\infty} e^{tA_\lambda}x = T(t)x \quad \text{for every} \quad x \in X. \tag{0.22}$$

The limit in (0.22) is uniform on bounded intervals. From (0.22) we can see that the limit $T(t)$ satisfies $T(0) = I$ and $\|T(t)\| \le 1$. Furthermore, the map $t \mapsto T(t)x$ is continuous for $t \ge 0$ as a uniform limit of the continuous function $t \to e^{tA_\lambda}x$. Thus $T(t)$ is a semigroup of contractions on X.

To end the proof we shall prove that A is the infinitesimal generator of $T(t)$.
Let $x \in D(A)$. Then by (0.22) and Theorem 2 it follows that

$$\begin{aligned} T(t)x - x &= \lim_{\lambda\to\infty} (e^{tA_\lambda}x - x) \\ &= \lim_{\lambda\to\infty} \int_0^t e^{tA_\lambda} A_\lambda x\, ds = \int_0^t T(s)Ax\, ds. \end{aligned} \tag{0.23}$$

where the last equality is deduced from the uniform convergence of $e^{tA_\lambda}A_\lambda x$ to $T(t)Ax$ on bounded intervals. Let now B be the infinitesimal generator of $T(t)$ and $x \in D(A)$. We observe from (0.23) that

$$\frac{T(t)x - x}{t} = \frac{1}{t}\int_0^t T(s)Ax\, ds \quad \text{for} \quad t > 0. \tag{0.24}$$

Then making $t \downarrow 0$ in (0.24) we get $x \in D(B)$ and $Bx = Ax$. Therefore $A \subseteq B$. Since B is the infinitesimal generator of $T(t)$, from the necessary conditions we deduce that $1 \in \rho(B)$. On the other hand, we assume that $1 \in \rho(A)$ by (ii) in the statement of the theorem. Since $A \subseteq B$, $(I - B)D(A) = (I - A)D(A) = X$ whence

$$D(B) = (I - B)^{-1}X = D(A).$$

Therefore $A = B$.

□

Corollary 3. *A linear operator* A *is the infinitesimal generator of a* C_0 *semigroup satisfying* $\|T(t)\| \leq e^{\omega t}$ *if and only if*

(i) A *is closed and* $\overline{D(A)} = X$*,*

(ii) The resolvent set satisfies that $\{\lambda : \mathrm{Im}\,\lambda = 0, \lambda > \omega\} \subset \rho(A)$ *and for such* λ

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda - \omega}.$$

Proof. Exercise. □

The Lumer-Phillips Theorem

In this section we present a characterization of infinitesimal generators of **dissipative** linear operators.

Definition 4. *Let X be a Banach space and let X^* be its dual. We denote the value of $x^* \in X^*$ at $x \in X$ by either $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$. For any $x \in X$ we define the* **duality set** *$F(x) \subseteq X^*$ by*

$$F(x) = \{x^* : x^* \in X^* \text{ and } \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}. \tag{0.25}$$

From the Hahn-Banach theorem we note that the set $F(x) \neq \emptyset$ for every $x \in X$.

Definition 5. *A linear operator A is* **dissipative** *if for every $x \in D(A)$ there exists $x^* \in F(x)$ such that*

$$\operatorname{Re}\langle Ax, x^*\rangle \le 0.$$

An explicit example of a dual set follows.

Example 1. *Let X be a Hilbert space with inner product $(\cdot|\cdot)$. We show that $F(x) = \{(\cdot|x)\}$.*

Indeed, for each $y^* \in X^*$ the Riesz' representation theorem gives a unique $y \in X$ such that $\langle x, y^* \rangle = (x|y)$ holds for all $x \in X$ and $\|y\| = \|y^*\|$. For an element $y^* \in F(x)$ we thus have $\|x\| = \|y\|$ and $(x|y) = \|x\|\|y\|$. The last equality holds if and only if x and y are linearly dependent. In view of the first equality there thus exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $x = \alpha y$. Hence, $y^* \in F(x)$ implies that $y = x$. The reverse implication is obvious.

Next we give a useful characterization of dissipative operators.

Theorem 8. *A linear operator* A *is dissipative if and only if*

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \quad \text{for all } x \in D(A) \text{ and } \lambda > 0. \tag{0.26}$$

Proof. Let A be dissipative, $\lambda > 0$ and $x \in D(A)$. If $x^* \in F(x)$ and $\mathrm{Re}\,\langle Ax, x^*\rangle \leq 0$ then

$$\|\lambda x - Ax\|\|x\| \geq |\langle \lambda x - Ax, x^*\rangle| \geq \mathrm{Re}\ \langle \lambda x - Ax, x^*\rangle \geq \lambda \|x\|^2$$

and (0.26) follows readily.

Conversely, let $x \in D(A)$ and assume that

$$\lambda\|x\| \leq \|\lambda x - Ax\| \text{ for all } \lambda > 0.$$

If $y_\lambda^* \in F(\lambda x - Ax)$ and $z_\lambda^* = \dfrac{y_\lambda^*}{\|y_\lambda^*\|}$ then $\|z_\lambda^*\| = 1$ and

$$\begin{aligned}\lambda\|x\| \leq \|\lambda x - Ax\| &= \langle \lambda x - Ax, z_\lambda^* \rangle \\ &= \lambda \mathrm{Re}\,\langle x, z_\lambda^* \rangle - \mathrm{Re}\,\langle Ax, z_\lambda^* \rangle \leq \lambda\|x\| - \mathrm{Re}\,\langle Ax, z_\lambda^* \rangle\end{aligned}$$

for every $\lambda > 0$. Therefore for $\lambda = n$ we have

$$\mathrm{Re}\,\langle Ax, z_n^* \rangle \leq 0 \quad \text{and} \quad \mathrm{Re}\,\langle x, z_n^* \rangle \geq \|x\| - \frac{1}{n}\|Ax\|. \qquad (0.27)$$

Since the unit ball of X^* is compact in the weak-star topology of X^* the sequence z_n^*, $n \to \infty$, has a weak-star limit point $z^* \in X^*$, $\|z^*\| = 1$.

From (0.27) it follows that $\mathrm{Re}\langle Ax, z^*\rangle \le 0$ and $\mathrm{Re}\langle x, z_n^*\rangle \ge \|x\|$. But

$$\mathrm{Re}\langle x, z_n^*\rangle \le |\langle x, z_n^*\rangle| \le \|x\|$$

and therefore $\langle x, z_n^*\rangle = \|x\|$. Taking $x^* = \|x\|\, z^*$ we have that

$$x^* \in F(x) \;\; \text{and} \;\; \mathrm{Re}\langle Ax, x^*\rangle \le 0.$$

Thus for every $x \in D(A)$ there is an $x^* \in F(x)$ such that

$$\mathrm{Re}\langle Ax, x^*\rangle \le 0$$

and A is dissipative. □

Theorem 9 (Lumer-Phillips). *Let A be a linear operator with dense domain $D(A)$ in X.*

(i) If A is dissipative and there is a $\lambda_0 > 0$ such that $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0 semigroup of contractions on X.

(ii) If A is the infinitesimal generator of a C_0 semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative. Moreover, for every $x \in D(A)$ and every $x^* \in F(x)$,

$$\mathrm{Re}\,\langle Ax, x^* \rangle \leq 0.$$

Proof. Let $\lambda > 0$, for A being dissipative we deduce from Theorem 8 that

$$\|(\lambda x - Ax)\| \geq \lambda \|x\| \quad \text{for every } \lambda > 0 \text{ and } x \in D(A). \tag{0.28}$$

Since $R(\lambda_0 I - A) = X$, (0.28) implies for $\lambda = \lambda_0$ that $(\lambda_0 I - A)^{-1}$ is a bounded linear operator and so it is closed. If $R(\lambda I - A) = X$ for every $\lambda > 0$ we have $\rho(A) \supseteq (0, \infty)$ and $\|R_\lambda(A)\| \leq \lambda^{-1}$ by (0.28). Thus the Hille-Yosida theorem yields that A is the infinitesimal generator of a C_0 semigroup of contractions on X.

To complete (i) it remains to show that $R(\lambda I - A) = X$ for all $\lambda > 0$. Consider

$$\Lambda = \{\lambda : 0 < \lambda < \infty \text{ and } R(\lambda I - A) = X\}.$$

Let $\lambda \in \Lambda$, from (0.28), $\lambda \in \rho(A)$. Since $\rho(A)$ is open, a neighborhood of λ is in $\rho(A)$. The intersection of this neighborhood with the real line is clearly in Λ and therefore Λ is open. On the other hand, let $\lambda_n \in \Lambda$, $\lambda_n \to \lambda > 0$. For every $y \in X$ there exists an $x_n \in D(A)$ such that

$$\lambda_n x_n - A x_n = y. \tag{0.29}$$

By using (0.28) we deduce that $\|x_n\| \leq \lambda_n^{-1} \|y\| \leq c$ for some $c > 0$. We notice now that

$$\begin{aligned} \lambda_m \|x_n - x_m\| &\leq \|\lambda_m(x_n - x_m) - A(x_n - x_m)\| \\ &= |\lambda_n - \lambda_m| \|x_n\| \\ &\leq c\,|\lambda_n - \lambda_m|. \end{aligned} \tag{0.30}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Let $x_n \to x$. Then by (0.29) $Ax_n \to \lambda x - y$. Since A is closed, $x \in D(A)$ and $\lambda x - Ax = y$. Therefore $R(\lambda I - A) = X$ and $\lambda \in \Lambda$. Thus Λ is closed in $(0, \infty)$ and since $\lambda_0 \in \Lambda$ by hypothesis $\Lambda \neq \emptyset$. Thus $\Lambda = (0, \infty)$. This completes the proof of (i).

Proof of (ii). If A is the infinitesimal generator of a C_0 semigroup of contractions, $T(t)$, on X, then by the Hille-Yosida theorem $\rho(A) \supseteq (0, \infty)$ and so $R(\lambda I - A) = X$ for all $\lambda > 0$. In addition, if $x \in D(A)$, $x^* \in F(x)$ then

$$|\langle T(t)x, x^* \rangle| \leq \|T(t)x\| \|x^*\| \leq \|x\|^2.$$

Therefore,

$$\mathrm{Re}\langle T(t)x - x, x^* \rangle = \mathrm{Re}\langle T(t)x, x^* \rangle - \|x\|^2 \leq 0 \qquad (0.31)$$

Dividing (0.31) by $t > 0$ and letting $t \downarrow 0$ yields

$$\mathrm{Re}\langle Ax, x^* \rangle \leq 0.$$

This holds for every $x^* \in F(x)$ and the proof is complete.

□

Corollary 4. *Let A a densely defined closed linear operator. If both A and A^* are dissipative, the operator A is the infinitesimal generator of a C_0 semigroup of contractions on X.*

Proof. From (i) in Theorem 9 it is sufficient to show that $R(I-A) = X$. Since A is dissipative and closed $R(I - A)$ is a closed subspace of X. If $R(I - A) \neq X$ then there exists $x^* \in X^*$, $x^* \neq 0$ such that $\langle x^*, x - Ax \rangle = 0$ for $x \in D(A)$. This implies $x^* - A^*x^* = 0$. Since A^* is also dissipative it follows from Theorem 8 that $x^* = 0$, contradicting the construction of x^*. □

Applications

In our first two example we will use the theory developed in [1] to study initial/boundary-value problem for second-order PDE. We will use a particular example.

Example 2. *We consider the initial/boundary-value problem*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T = U \times (0, T), \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}, \end{cases} \tag{0.32}$$

we suppose that the bounded open set U has a smooth boundary.

We will reinterpret (0.32) as the flow determined by a semigroup on $X = L^2(U)$. We set

$$D(A) = H^1_0(U) \cap H^2(U), \tag{0.33}$$

and define

$$Au = \Delta u \text{ if } u \in D(A). \tag{0.34}$$

We already saw that A is an unbounded linear operator on X.

□

We define the bilinear form associated to $-\Delta$, as

$$B[u,v] = \int_U \nabla u \cdot \nabla v \, dx. \tag{0.35}$$

Using the Poincaré inequality

$$\|u\|^2_{L^2(U)} \leq \|\nabla u\|^2_{L^2(U)} \tag{0.36}$$

it follows that

$$\frac{1}{2}\|u\|^2_{H^1_0(U)} \leq B[u,u]. \tag{0.37}$$

Claim: The operator A generates a contraction semigroup $\{T(t)\}_{t\geq 0}$ in $L^2(U)$.

We first notice that $D(A)$ given by (0.33) is dense in $L^2(U)$. The operator A is closed. Indeed, let $\{u_j\}_{j=1}^\infty \subset D(A)$ with

$$u_j \to u, \quad Au_j \to v \ \text{ in } \ L^2(U) \tag{0.38}$$

By the regularity theory (see Theorem 4 Section 6.3.2 in [1])

$$\|u_j - u_k\|_{H^2(U)} \leq c\,\|Au_j - Au_k\|_{L^2(U)} \tag{0.39}$$

for all j, k. Thus (0.39) implies $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $H^2(U)$ and so

$$u_j \to u \ \text{ in } \ H^2(U). \tag{0.40}$$

Therefore $u \in D(A)$. Furthermore (0.40) implies that $\Delta u_j \to \Delta u$ in $L^2(U)$, and so $v = Au$.

Formally, assuming u smooth and vanishing rapidly as $|x| \to \infty$ we can obtain (0.39) in our case. Suppose u is a solution of $-\Delta u = f$, then integrating by parts twice we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f^2\, dx = \int_{\mathbb{R}^n} (\Delta u)^2\, dx &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} u_{x_i x_i} u_{x_j x_j}\, dx \\ &= -\sum_{i,j=1}^{n} \int_{\mathbb{R}^n} u_{x_i x_i x_j} u_{x_j}\, dx \\ &= \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} u_{x_i x_j} u_{x_j x_i}\, dx \\ &= \int_{\mathbb{R}^n} |D^2 u|^2\, dx. \end{aligned}$$

Regularizing u and using that $u \in H_0^1(U)$ the argument above yields inequality (0.39).

Next we check the resolvent conditions.

From the Fredholm theory (see Theorem 3 in §6.2.2 in [1] for a second order elliptic operator) for $\lambda > 0$ the BVP

$$\begin{cases} -\Delta u + \lambda u = f \ \text{ in } \ U, \\ \qquad u = 0 \qquad\quad \text{ in } \ \partial U, \end{cases} \tag{0.41}$$

has a unique weak solution $u \in H_0^1(U)$ for each $f \in L^2(U)$, i.e. there is a unique $u \in H_0^1(U)$ such that

$$B[u, v] + \lambda(u, v) = (f, v) \ \text{ for all } \ v \in H_0^1(U), \tag{0.42}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(U)$. By the regularity theory (see (0.39)) $u \in H^2(U) \cap H_0^1(U)$. Hence $u \in D(A)$. Now we write (0.41) as

$$\lambda u - Au = f \tag{0.43}$$

Thus $(\lambda I - A) : D(A) \to X$ is one-to-one and onto, provided $\lambda > 0$. Hence $(0, \infty) \subset \rho$.

Consider the weak form of the BVP (0.41), (0.42) and setting $v = u$ we have

$$\lambda \|u\|^2_{L^2(U)} \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)}.$$

Hence, since $u = R_\lambda f$, we have the estimate

$$\|R_\lambda f\|_{L^2(U)} \leq \frac{1}{\lambda} \|f\|_{L^2(U)}.$$

This bound is valid for all $f \in L^2(U)$. Thus

$$\|R_\lambda\| \leq \frac{1}{\lambda}.$$

Collecting the previous information we can apply the Hille-Yosida theorem to prove our claim.

Example 3. *We consider the initial/boundary-value problem associated to the wave equation,*

$$\begin{cases} u_{tt} - \Delta u = 0 \ \text{ in } \ U_T = U \times [0,T] \\ u = 0 \ \text{ on } \ \partial U \times [0,T] \\ u = g, \ u_t = h \ \text{ on } \ U \times \{t = 0\} \end{cases} \tag{0.44}$$

where U is a bounded open set in $\mathbb{R}^n$ with smooth boundary.
We rewrite (0.44) *as a first order system by letting $v = u_t$, that is,*

$$\begin{cases} u_t = v, \ \ v_t - \Delta u = 0 \ \text{ in } \ U_T = U \times [0,T] \\ u = 0 \ \text{ on } \ \partial U \times [0,T] \\ u = g, \ u_t = h \ \text{ on } \ U \times \{t = 0\}. \end{cases} \tag{0.45}$$

Using the Poincaré inequality $\|u\|^2_{L^2(U)} \leq c\|\nabla u\|^2_{L^2(U)}$, we have that

$$\frac{1}{2}\|u\|^2_{H^1_0(U)} \leq \|\nabla u\|^2_{L^2(U)}. \tag{0.46}$$

We take $X = H^1_0(U) \times L^2(U)$ with the norm

$$\|(u,v)\| = \left(\|\nabla u\|^2_{L^2(U)} + \|v\|^2_{L^2(U)}\right)^{1/2}.$$

Define

$$D(A) = [H^2(U) \cap H^1_0(U)] \times H^1_0(U)$$

and

$$A(u,v) = (v, \Delta u) \ \text{ for } \ (u,v) \in D(A). \tag{0.47}$$

□

We shall show that A satisfies the hypothesis of the Hille-Yosida theorem.

1. The domain of A is dense in $H_0^1(U) \times L^2(U)$.

2. A is closed. Indeed, let $\{(u_k, v_k)\}_{k=1}^{\infty} \subset D(A)$ such that

$$(u_k, v_k) \to (u, v), \quad A(u_k, v_k) \to (f, g) \ \text{ in } \ H_0^1(U) \times L^2(U).$$

Since $A(u_k, v_k) = (v_k, \Delta u_k)$, we have that $f = v$ and $-\Delta u_k = -g$ in $L^2(U)$. By the regularity theory (see Theorem 4 Section 6.3.2 in [1])

$$\|u_j - u_k\|_{H^2(U)} \leq c \, \|\Delta u_j - \Delta u_k\|_{L^2(U)} \tag{0.48}$$

for all j, k. It follows that $u_k \to u$ in $H^2(U)$ and $g = \Delta u$. Thus $(u, v) \in D(A)$, $A(u, v) = (v, \Delta u) = (f, g)$

3. Now let $\lambda > 0$, $(f, g) \in X = H_0^1(U) \times L^2(U)$, and consider the operator equation

$$\lambda(u, v) - A(u, v) = (f, g). \tag{0.49}$$

or equivalently

$$\begin{cases} \lambda u - v = f & u \in H^2(U) \cap H_0^1(U) \\ \lambda v - \Delta u = g & v \in H_0^1(U). \end{cases} \tag{0.50}$$

But (0.50) implies

$$\lambda^2 u - \Delta u = \lambda f + g, \quad u \in H^2(U) \cap H_0^1(U). \tag{0.51}$$

Since $\lambda^2 > 0$, estimate (0.46) and the regularity theory imply there exists a unique solution u of (0.51). Defining $v = \lambda u - f$ in $H_0^1(U)$ we have proved that (0.49) has a unique solution (u, v). Thus $(0, \infty) \subset \rho(A)$.

4. Whenever (0.49) holds, we write $(u, v) = R_\lambda(f, g)$. Now from the second equation in (0.50) we deduce

$$\lambda \|v\|_{L^2(U)}^2 + \int_U \nabla u \cdot \nabla v \, dx = \int_U g v \, dx.$$

Substituting $v = \lambda u - f$, we obtain

$$\begin{aligned} \lambda\Big(\|v\|_{L^2(U)}^2 + \int_U \nabla u \cdot \nabla u \, dx\Big) &= \int_U g v \, dx + \int_U \nabla u \cdot \nabla f \, dx \\ &\leq (\|g\|_{L^2(U)}^2 + \|\nabla f\|_{L^2(U)}^2)^{1/2} (\|v\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2)^{1/2}. \end{aligned}$$

From the definition

$$\|(u,v)\| \leq \frac{1}{\lambda}\|(f,g)\|$$

and thus

$$\|R_\lambda\| \leq \frac{1}{\lambda}.$$

We verified the hypotheses of the Hille-Yosida theorem which guarantees the existence of a C_0 semigroup of operators associated to IBVP (0.45).

□

Example 4. *Consider the linear operator*

$$A : H^4(\mathbb{R}) \to L^2(\mathbb{R})$$

defined by

$$u \mapsto -(1 - \partial_x^2)^2 u.$$

We will show that A generates a C_0-semigroup on $L^2(\mathbb{R})$ by using the Hille-Yosida theorem.

In general, Fourier transforms are used in this context (when allowed) to prove existence and uniqueness of solution for the resolvent equation $(\lambda - A)u = f$, which is needed in the application of the Hille-Yosida Theorem (for all $\lambda > 0$) or Lumer-Phillips Theorem (for some $\lambda > 0$).

Here the operator $A : D(A) \subset X \to X$ is defined by

$$D(A) = H^4(\mathbb{R}), \quad X = L^2(\mathbb{R}), \quad Au = -u + 2u_{xx} - u_{xxxx}.$$

Fixed $\lambda > 0$, we have to show that: given $f \in L^2(\mathbb{R})$, there exists a unique $u \in H^4(\mathbb{R})$ such that

$$\lambda u + u - 2u_{xx} + u_{xxxx} = f \tag{0.52}$$

Uniqueness: Let u be the solution in $H^4(\mathbb{R})$ of (0.52). Then taking Fourier transform, we conclude that

$$u = \Big(\frac{\widehat{f}}{\lambda + 1 + 2\xi^2 + \xi^4}\Big)^{\vee}. \tag{0.53}$$

This show that (0.52) have at most one solution in $H^4(\mathbb{R})$.

Existence: Define

$$u = \Big(\frac{\widehat{f}}{\lambda + 1 + 2\xi^2 + \xi^4}\Big)^{\vee}.$$

Since $\widehat{f} \in L^2(\mathbb{R})$ and

$$\Big|\frac{\widehat{f}}{\lambda + 1 + 2\xi^2 + \xi^4}\Big| \leq |\widehat{f}|.$$

we have that $\dfrac{\widehat{f}}{\lambda + 1 + 2\xi^2 + \xi^4} \in L^2(\mathbb{R})$. Thus u is well defined and belongs to $L^2(\mathbb{R})$. Taking Fourier transform in (0.53), we obtain

$$(\lambda + 1 + 2\xi^2 + \xi^4)\widehat{u} = \widehat{f} \in L^2. \tag{0.54}$$

and then $(1+\xi^4)\widehat{u} \in L^2$ which implies that $u \in H^4(\mathbb{R})$. From (0.54),

$$(\lambda + 1 - 2(i\xi)^2 + (i\xi)^4)\widehat{u} = \widehat{f}$$

and then, since $u \in H^4(\mathbb{R})$, it follows that

$$\lambda u + u - 2u_{xx} + u_{xxxx} = f.$$

This shows that (0.52) have a solution in $H^4(\mathbb{R})$.

From (0.53)

$$\|(\lambda - A)^{-1} f\| = \|u\| = \|\widehat{u}\| = \left\| \frac{\widehat{f}}{\lambda + 1 + 2\xi^2 + \xi^4} \right\| \leq \frac{1}{\lambda} \|\widehat{f}\| = \frac{1}{\lambda} \|f\|$$

and thus

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda},$$

which is an estimate we need to apply the Hille-Yosida theorem. □

Remark 1. *We can use Fourier transform to find the candidate of C_0 semigroup in the previous example. Indeed it should be*

$$T(t)u = (e^{-t(1+\xi^2)^2} \widehat{u})^{\vee}, \quad t \geq 0,$$

where $\widehat{\ }$, $^{\vee}$ *are the Fourier transform and inverse Fourier transform. Verify that $T(t)$ is indeed a C_0 semigroup.*

Example 5. *Define for every* $v \in Y = H^s(\mathbb{R})$, $s \geq 3$, *an operator* $A_1(v)$ *by* $D(A_1(v)) = H^1(\mathbb{R})$ *and for* $u \in D(A_1(v))$, $A_1(v)u = vDu$.

There exists $\beta > 0$ *such that* $-(A_1(v) + \beta I)$ *is a dissipative operator.*

First we note that since $v \in H^s(\mathbb{R})$, $Dv \in H^{s-1}(\mathbb{R})$. *Since* $s \geq 3$ *the Sobolev embedding guarantees that* $Dv \in L^\infty(\mathbb{R})$ *and*

$$\|Dv\|_\infty \leq c\|v\|_{H^{s-1}} \leq c\,\|v\|_{H^s}.$$

Now, for every $u \in H^1(\mathbb{R})$ we have

$$(A_1(v)u, u) = \int v\, Du \cdot u\, dx = \frac{1}{2} \int v Du^2\, dx = -\int Dv\, u^2\, dx$$
$$\geq -\frac{1}{2} \|Dv\|_\infty \|u\|_2^2 \geq -c_0\, \|v\|_{H^s} \|u\|_2^2.$$

Taking $\beta > \beta_0(v) = c_0 \|v\|_{H^s}$ we have the result.

Example 6. *Consider the Laplace operator with Dirichlet boundary conditions. Let $X = L^2(0,\pi)$ and consider the operator*

$$(Af)(x) = f''(x)$$

with domain

$$D(A) = \{f \in L^2(0,\pi) : f \text{ cont. differentiable on } [0,\pi], f'' \text{ exists a.e.}$$
$$f'' \in L^2(0,\pi),\ f'(x) - f'(0) = \int_0^x f''(s)\,ds \text{ for } x \in [0,\pi]$$
$$\text{and } f(0) = f(\pi) = 0\}$$

Then

$$\langle Af, f\rangle = \int_0^\pi f''(s)\overline{f(s)}\,ds = -\int f'(s)\overline{f'(s)}\,ds = -\|f'\|^2 \le 0.$$

Example 7. *Let* $X = L^2(\mathbb{R})$ *and* $Af = f'$ *with* $D(A) = C_0^1(\mathbb{R})$*. Then*

$$\langle Af, f\rangle = \int_{\mathbb{R}} f' \cdot \bar{f} = -\int_{\mathbb{R}} f \cdot \overline{f'} = -\langle f, Af\rangle = -\overline{\langle Af, f\rangle}$$

for $f \in D(A)$*, showing that*

$$\langle Af, f\rangle + \overline{\langle Af, f\rangle} = 0, \quad \text{i.e} \quad \langle Af, f\rangle \in i\mathbb{R}.$$

This means that both A *and* $-A$ *are dissipative.*

Exercises

1. Consider the initial value problem for the linear hyperbolic system

$$\begin{cases} \overrightarrow{u}_t + A(x,t)\overrightarrow{u}_x = D(x,t)\overrightarrow{u} + \overrightarrow{f}(x,t) \text{ for } x \in \mathbb{R},\ 0 < t < T, \\ \overrightarrow{u}(x,0) = \overrightarrow{g}(x) \text{ for } x \in \mathbb{R}. \end{cases}$$

   Suppose the matrices A, D and $\overrightarrow{g}$, $\overrightarrow{f}$ are sufficiently regular. Prove that this problem admits a unique classical solution $\overrightarrow{u} \in C_B^1(\mathbb{R} \times [0,T]; \mathbb{R}^n)$.

2. Show that a semigroup of operators in a Banach space is strongly continuous in $[0, \infty)$, i.e. $S(t)\phi \to S(t_0)\phi$ when $t \to t_0$, if the continuity is satisfied at $t_0 = 0$.

3. Let $\varphi(x)$ be a function defined in $-\infty < x < \infty$. Let $(S(t)\varphi)(x) = \varphi(x+t)$, clearly $\{S(t)\}_{t\geq 0}$ satisfies the first two semigroup properties.

   (a) Is $S(t)$ strongly continuous in $X = L^2(\mathbb{R}^n)$?

   (b) Is $S(t)$ is strongly continuous in $X = C_B(\mathbb{R}^n)$? Where $C_B(\mathbb{R}^n)$ denotes the space of bounded continuous functions in $\mathbb{R}^n$.

4. Define for $t > 0$

$$(S(t)g)(x) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy, \quad x \in \mathbb{R}^n$$

   where $g : \mathbb{R}^n \to \mathbb{R}$ and Φ is the fundamental solution of the heat equation. Set $S(0)g = g$.

   (a) Prove that $\{S(t)\}_{t\geq 0}$ is a semigroup of contractions in $L^2(\mathbb{R}^n)$.

   (b) Show that $\{S(t)\}_{t\geq 0}$ is not a semigroup of contractions in $L^\infty(\mathbb{R}^n)$.

5. (a) Prove that the infinitesimal generator of a C_0-semigroup in X is a closed operator.

   (b) Suppose $A : D \to X$ is a closed operator and $\varphi \in C([0,T];X)$ satisfies $\varphi(t) \in D$ for all $t \in [0,T]$. Show that $\varphi \in C([0,T];D)$ if and only if $A\varphi \in C([0,T];X)$.

6. Let X be a Banach space and $f \in C([0,T];X)$.

   (a) Show that the Riemann integral $F(t) = \int_0^t f(s)\,ds$ exists for $0 \le t \le T$.

   (b) Prove that $F \in C([0,T];X)$ and $f(0) = \lim_{t\to 0^+} t^{-1} F(t)$.

7. Let X be a Banach space, if $f \in C([0,T];X)$ and $S(t)$ is a C_0-semigroup, show that $h(s) = S(t-s)f(s) \in C([0,T];X)$ for all $0 \le t \le T$.

8. Let $A : D \to X$ be a closed operator in a Banach space X and $f \in C([0,T];D)$. Let $u(t) = \int_0^t f(s)\,ds$. Prove that $u \in C([0,T];D)$ and $Au(t) = \int_0^t Af(s)\,ds$.

9. Let $\{S(t)\}_{t\geq 0}$ be a semigroup of contractions in X, with infinitesimal generator A. Inductively define $D(A^k) = \{x \in D(A^{k-1}) : A^{k-1}x \in D(A)\}$, $k = 2, \ldots$. Show that if $x \in D(A^k)$, for some k, then $S(t)x \in D(A^k)$ for all $t \geq 0$.

10. Use the previous exercise to prove that if u is a solution in $X = L^2(U)$ of

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{em } U_T \\ u = 0 & \text{em } \partial U \times [0,T] \\ u = g & \text{em } U \times \{t = 0\}, \end{cases}$$

with $g \in C_c^\infty(U)$, then $u(\cdot, t) \in C^\infty(U)$ for each $0 \leq t \leq T$.

11. Show that a linear operator A is dissipative if and only if

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \text{ for all } x \in D(A) \text{ and } \lambda > 0.$$

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