Teoria Espectral

The Fourier Transform in $L^1(\mathbb{R}^n)$

Definition 1. *The Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$, denoted by $\widehat{f}$, is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x \cdot \xi)} dx, \quad \text{for } \xi \in \mathbb{R}^n, \tag{0.1}$$

where $(x \cdot \xi) = x_1 \xi_1 + \cdots + x_n \xi_n$.

We list some basic properties of the Fourier transform in $L^1(\mathbb{R}^n)$.

Theorem 1. *Let* $f \in L^1(\mathbb{R}^n)$. *Then:*

1. $f \mapsto \widehat{f}$ *defines a linear transformation from* $L^1(\mathbb{R}^n)$ *into* $L^\infty(\mathbb{R}^n)$ *with*

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{0.2}$$

2. $\widehat{f}$ *is continuous.*

3. $\widehat{f}(\xi) \to 0$ *as* $|\xi| \to \infty$ *(Riemann–Lebesgue).*

4. *If* $\tau_h f(x) = f(x-h)$ *denotes the translation by* $h \in \mathbb{R}^n$, *then*

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi i(h\cdot\xi)}\widehat{f}(\xi), \tag{0.3}$$

and

$$(\widehat{e^{-2\pi i(x\cdot h)}f})(\xi) = (\tau_{-h}\widehat{f}\,)(\xi). \tag{0.4}$$

5. *If* $\delta_a f(x) = f(ax)$ *denotes a dilation by* $a > 0$, *then*

$$\widehat{(\delta_a f)}(\xi) = a^{-n}\widehat{f}(a^{-1}\xi). \tag{0.5}$$

6. *Let* $g \in L^1(\mathbb{R}^n)$ *and* $f * g$ *be the convolution of* f *and* g. *Then*

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{0.6}$$

7. *Let* $g \in L^1(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy. \tag{0.7}$$

Notice that the equality in (0.2) holds for $f \geq 0$, i.e.,

$$\widehat{f}(0) = \|\widehat{f}\|_\infty = \|f\|_1.$$

Proof. It is left as an exercise. $\square$ $\square$

Next we give some examples to illustrate the properties stated in Theorem 1.

Example 1. *Let* $n = 1$ *and* $f(x) = \chi_{(a,b)}(x)$ *(the characteristic function of the interval* (a, b)*). Then*

$$\begin{aligned}\widehat{f}(\xi) &= \int_a^b e^{-2\pi i x\xi}\,dx \\ &= -\frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{2\pi i\xi} \\ &= -e^{-\pi i(a+b)\xi}\,\frac{\sin(\pi(a-b)\xi)}{\pi\xi}.\end{aligned}$$

Notice that $\widehat{f} \notin L^1(\mathbb{R})$ and that $\widehat{f}(\xi)$ has an analytic extension $\widehat{f}(\xi+i\eta)$ to the whole plane $\xi + i\eta \in \mathbb{C}$. In particular, if $(a,b) = (-k,k)$, $k \in \mathbb{Z}^+$, we have

$$\widehat{\chi}_{(-k,k)}(\xi) = \frac{\sin(2\pi k\xi)}{\pi\xi}.$$

$\square$

Example 2. *Let* $n = 1$ *and for* $k \in \mathbb{Z}^+$ *define*

$$g_k(x) = \begin{cases} k+1+x, & \text{if } x \in (-k-1, -k+1] \\ 2, & \text{if } x \in (-k+1, k-1) \\ k+1-x, & \text{if } x \in [k-1, k+1) \\ 0, & \text{if } x \notin (-k-1, k+1), \end{cases}$$

i.e., $g_k(x) = \chi_{(-1,1)} * \chi_{(-k,k)}(x)$. *The identity* (0.6) *and the previous example show that*

$$\widehat{g}_k(\xi) = \frac{\sin(2\pi\xi)\sin(2\pi k\xi)}{(\pi\xi)^2}.$$

Notice that $\widehat{g}_k \in L^1(\mathbb{R})$ *and has an analytic extension to the whole plane* $\mathbb{C}$.

Example 3. *Let* $n \geq 1$ *and* $f(x) = e^{-4\pi^2 t|x|^2}$ *with* $t > 0$. *Then changing variables* $x \to x/\sqrt{t}$ *and using* (0.5), *we can restrict ourselves to the case* $t = 1$. *From Fubini's theorem we write*

$$\begin{aligned}
\int\limits_{\mathbb{R}^n} e^{-4\pi^2|x|^2} e^{-2\pi i(x\cdot\xi)}\, dx &= \prod_{j=1}^{n} \int\limits_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i \xi_j x_j)}\, dx_j \\
&= \prod_{j=1}^{n} \int\limits_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i \xi_j x_j + \xi_j^2/4)} e^{-\xi_j^2/4}\, dx_j \\
&= \prod_{j=1}^{n} e^{-\xi_j^2/4} \int\limits_{-\infty}^{\infty} e^{-(2\pi x_j + i\xi_j/2)^2}\, dx_j \\
&= 2^{-n} \pi^{-n/2} e^{-|\xi|^2/4},
\end{aligned}$$

where in the last equality we have employed the following identities from complex integration and calculus,

$$\int_{-\infty}^{\infty} e^{-(2\pi x+i\xi/2)^2}\,dx = \int_{-\infty}^{\infty} e^{-(2\pi x)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,\frac{dx}{2\pi} = \frac{1}{2\sqrt{\pi}}.$$

Hence

$$\widehat{e^{-4\pi^2 t|x|^2}}(\xi) = \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{n/2}}. \tag{0.8}$$

Observe that taking $t = 1/4\pi$ and changing variables $t \to 1/16\pi^2 t$ we get

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2} \quad \text{and} \quad \widehat{\frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}}(\xi) = e^{-4\pi^2 t|\xi|^2},$$

respectively.

$\square$

Example 4. *Let* $n \geq 1$ *and* $f(x) = e^{-2\pi|x|}$*. Then*

$$\hat{f}(\xi) = \frac{\Gamma[\frac{(n+1)}{2}]}{\pi^{(n+1)/2}} \frac{1}{(1 + |\xi|^2)^{(n+1)/2}},$$

where $\Gamma(\cdot)$ *denotes the Gamma function. See Exercise 1.1 (i).*

Example 5. *Let* $n = 1$ *and* $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. *Using complex integration one obtains the identity*

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{b}\,e^{-ab}, \quad a,\ b > 0.$$

Hence

$$\frac{1}{\pi}\widehat{\frac{1}{1+x^2}}(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{1+x^2}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(2\pi|\xi|x)}{1+x^2}\,dx = e^{-2\pi|\xi|}.$$

One of the most important features of the Fourier transform is its relationship with differentiation. This is described in the following results.

Proposition 1. *Suppose* $x_k f \in L^1(\mathbb{R}^n)$*, where* x_k *denotes the* k*th coordinate of* x*. Then* $\widehat{f}$ *is differentiable with respect to* ξ_k *and*

$$\frac{\partial \widehat{f}}{\partial \xi_k}(\xi) = (-2\pi i \widehat{x_k f}(x))(\xi). \tag{0.9}$$

In other words, the Fourier transform of the product $x_k f(x)$ is equal to a multiple of the partial derivative of $\widehat{f}(\xi)$ with respect to the kth variable.

To consider the converse result we need to introduce a definition.

Definition 2. Let $1 \le p < \infty$. A function $f \in L^p(\mathbb{R}^n)$ is *differentiable in* $L^p(\mathbb{R}^n)$ *with respect to the* k*th variable* if there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\int\limits_{\mathbb{R}^n} \left| \frac{f(x + he_k) - f(x)}{h} - g(x) \right|^p dx \to 0 \quad \text{as} \quad h \to 0,$$

where e_k has kth coordinate equals 1 and 0 in the others. If such a function g exists (in this case it is unique) it is called *the partial derivative of* f *with respect to the* k*th variable in the* L^p*-norm.*

Theorem 2. *Let* $f \in L^1(\mathbb{R}^n)$ *and* g *be its partial derivative with respect to the* k*th variable in the* L^1*-norm. Then* $\widehat{g}(\xi) = 2\pi i \xi_k \widehat{f}(\xi)$*.*

Proof. Properties (0.2) and (0.4) in Theorem 1 allow us to write

$$\left|\widehat{g}(\xi) - \widehat{f}(\xi)\frac{(1 - e^{-2\pi i h(\xi \cdot e_k)})}{h}\right|,$$

then take $h \to 0$ to obtain the result. $\square$

From the previous theorems it is easy to obtain the formulae

$$
\begin{aligned}
P(D)\widehat{f}(\xi) &= (P(-2\pi i x)f(x))^{\wedge}(\xi),\\
(\widehat{P(D)f})(\xi) &= P(2\pi i\xi)\widehat{f}(\xi),
\end{aligned}
\tag{0.10}
$$

where P is a polynomial in n variables and $P(D)$ denotes the differential operator associated to P.

Now we turn our attention to the following question: Given the Fourier transform $\widehat{f}$ of a function in $L^1(\mathbb{R}^n)$, how can we recover f ?

Examples 3–5 suggest the use of the formula

$$f(x) = \int\limits_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi)} d\xi.$$

Unfortunately, $\widehat{f}(\xi)$ may be nonintegrable (see Example 1). To avoid this problem one needs to use the so called "method of summability" (Abel and Gauss) similar to those used in the study of Fourier series.

Combining the idea behind the Gauss summation method and the identities (0.4), (0.7), (0.8) we obtain the following equalities:

$$
\begin{aligned}
f(x) &= \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x) = \lim_{t\to 0} \int\limits_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\
&= \lim_{t\to 0} \int\limits_{\mathbb{R}^n} \tau_x \frac{e^{-|y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\
&= \lim_{t\to 0} \int\limits_{\mathbb{R}^n} (\widehat{e^{2\pi i(x\cdot\xi)} e^{-4\pi^2 t|\xi|^2}})(y) f(y)dy \\
&= \lim_{t\to 0} \int\limits_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi)d\xi,
\end{aligned}
$$

where the limit is taken in the L^1-norm.

Thus, if f and $\widehat{f}$ are both integrable the Lebesgue dominated convergence theorem guarantees the pointwise equality. Also if $f \in L^1(\mathbb{R}^n)$ is continuous at the point x_0 we get

$$f(x_0) = \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x_0) = \lim_{t\to 0} \int\limits_{\mathbb{R}^n} e^{2\pi i(x_0\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi) d\xi.$$

Collecting this information we get the following result.

Proposition 2. *Let* $f \in L^1(\mathbb{R}^n)$*. Then*

$$f(x) = \lim_{t \to 0} \int\limits_{\mathbb{R}^n} e^{2\pi i (x \cdot \xi)} e^{-4\pi^2 t |\xi|^2} \widehat{f}(\xi) d\xi,$$

where the limit is taken in the L^1*-norm. Moreover, if* f *is continuous at the point* x_0 *then the following pointwise equality holds*

$$f(x_0) = \lim_{t \to 0} \int\limits_{\mathbb{R}^n} e^{2\pi i (x_0 \cdot \xi)} e^{-4\pi^2 t |\xi|^2} \widehat{f}(\xi) d\xi.$$

Let $f,\ \widehat{f} \in L^1(\mathbb{R}^n)$*. Then*

$$f(x) = \int\limits_{\mathbb{R}^n} e^{2\pi i (x \cdot \xi)} \widehat{f}(\xi) d\xi, \quad \text{almost everywhere } x \in \mathbb{R}^n.$$

From this result and Theorem 1 we can conclude that

$$\wedge \ : \ L^1(\mathbb{R}^n) \longrightarrow C_\infty(\mathbb{R}^n)$$

is a linear, one-to-one (Exercise 1.6 (i)), bounded map. However it is not surjective (Exercise 1.6 (iii)).

The Fourier Transform in $L^2(\mathbb{R}^n)$

To define the Fourier transform in $L^2(\mathbb{R}^n)$ we first shall use that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a dense subset of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Theorem 3 (Plancherel)**.** *Let* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$*. Then* $\widehat{f} \in L^2(\mathbb{R}^n)$ *and*

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{0.11}$$

Proof. Let $g(x) = \bar{f}(-x)$. Using Young's inequality, (0.6), and Exercise 1.7 (ii), it follows that

$$f * g \in L^1(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n) \quad \text{and} \quad \widehat{(f * g)}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi).$$

Since $\widehat{g} = \overline{(\widehat{f}\,)}$ we find that $\widehat{(f * g)} = |\widehat{f}\,|^2 \geq 0$.

Hence $\widehat{(f * g)} \in L^1(\mathbb{R}^n)$ (see Exercise 1.7 (iii)).

Proposition 2 shows that

$$(f * g)(0) = \int_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi,$$

$\square$

and

$$
\begin{aligned}
\|\widehat{f}\,\|_2^2 &= \int\limits_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi = (f * g)(0) \\
&= \int\limits_{\mathbb{R}^n} f(x) g(0 - x)\, dx = \int\limits_{\mathbb{R}^n} f(x) \bar{f}(x)\, dx = \|f\|_2^2.
\end{aligned}
$$

$\square$

This result shows that the Fourier transform defines a linear bounded operator from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Indeed, this operator is an isometry. Thus, there is a unique bounded extension $\mathcal{F}$ defined in all $L^2(\mathbb{R}^n)$. $\mathcal{F}$ is called the Fourier transform in $L^2(\mathbb{R}^n)$. We shall use the notation $\widehat{f} = \mathcal{F}(f)$ for $f \in L^2(\mathbb{R}^n)$. In general, the definition $\widehat{f}$ is realized as a limit in L^2 of the sequence $\{\widehat{h}_j\}$, where $\{h_j\}$ denotes any sequence in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that converges to f in the L^2-norm.

It is convenient to take

$$h_j(x) = \begin{cases} f(x), & \text{for } |x| \le j, \\ 0, & \text{for } |x| > j. \end{cases}$$

Then,

$$\widehat{h}_j(\xi) = \int\limits_{|x|<j} f(x) e^{-2\pi i (x\cdot\xi)}\, dx = \int\limits_{\mathbb{R}^n} h_j(x) e^{-2\pi i (x\cdot\xi)}\, dx$$

and so

$$\widehat{h}_j(\xi) \to \widehat{f}(\xi) \quad \text{in } L^2 \text{ as } j \to \infty.$$

Example 6. *Let* $n = 1$ *and* $f(x) = \dfrac{1}{\pi}\dfrac{x}{1+x^2}$. *Observe that* $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. *Differentiating the identity in the Example 5 with respect to* a *and taking* $b = 1$ *we get*

$$\int_{-\infty}^{\infty} \frac{x\ \sin(ax)}{1+x^2}\,dx = \pi e^{-a}, \qquad a > 0,$$

which combined with the previous remark gives

$$\widehat{f}(\xi) = -i\ \textit{sgn}(\xi) e^{-2\pi|\xi|}.$$

A surjective isometry defines a “unitary operator." Theorem 3 affirms that $\mathcal{F}$ is an isometry. Let us see that $\mathcal{F}$ is also surjective.

Theorem 4. *The Fourier transform defines a unitary operator in* $L^2(\mathbb{R}^n)$*.*

Proof. From the identity (0.11) it follows that $\mathcal{F}$ is an isometry. In particular, its image is a closed subspace of $L^2(\mathbb{R}^n)$. Assume that this is a proper subspace of L^2. Then there exists $g \neq 0$ such that

$$\int_{\mathbb{R}^n} \widehat{f}(y) g(y) dy = 0, \quad \text{for any } f \in L^2(\mathbb{R}^n).$$

□

Using formula (0.7) (Theorem 1), which obviously extends to $f, g \in L^2(\mathbb{R}^n)$, we have that

$$\int\limits_{\mathbb{R}^n} f(y)\widehat{g}(y)dy = \int\limits_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = 0, \quad \text{for any } f \in L^2.$$

Therefore $\widehat{g}(\xi) = 0$ almost everywhere, which contradicts

$$\|g\|_2 = \|\widehat{g}\|_2 \neq 0.$$

$\square$

Theorem 5. *The inverse of the Fourier transform* $\mathcal{F}^{-1}$ *can be defined by the formula*

$$\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x), \quad \text{for any} \quad f \in L^2(\mathbb{R}^n). \tag{0.12}$$

Proof. $\mathcal{F}^{-1}\widehat{f} = \tilde{f}$ is the limit in the L^2-norm of the sequence

$$f_j(x) = \int\limits_{|\xi|<j} \widehat{f}(\xi)e^{2\pi i(\xi\cdot x)}d\xi.$$

$\square$

First, we consider the case where $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. It suffices to verify this agrees with $\mathcal{F}^* \widehat{f}$, where $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ (we recall the fact that for a unitary operator the adjoint and the inverse are equal). This can be checked as follows:

$$\tilde{f}(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i(\xi \cdot x)} d\xi = \lim_{j \to \infty} f_j(x) \quad \text{in} \quad L^2(\mathbb{R}^n),$$

and

$$(g, \tilde{f}) = \int\limits_{\mathbb{R}^n} g(x) \overline{\Big(\int\limits_{\mathbb{R}^n} \widehat{f}(\xi)\, e^{2\pi i(\xi \cdot x)}\, d\xi \Big)} dx$$

$$= \int\limits_{\mathbb{R}^n} \Big(\int\limits_{\mathbb{R}^n} g(x) e^{-2\pi i(x\cdot\xi)} dx \Big) \overline{\widehat{f}(\xi)} d\xi = (\mathcal{F}g, \widehat{f})$$

for any $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Hence $\tilde{f} = f$.

The general case follows by combining the above result and an argument involving a justification of passing to the limit.

$\square$

Tempered Distributions

From the definitions of the Fourier transform on $L^1(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$ there is a natural extension to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. It is not hard to see that $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ contains the spaces $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$. On the other hand, as we shall prove, any function in $L^p(\mathbb{R}^n)$ for $p > 2$ has a Fourier transform in the distribution sense. However, they may not be functions they are tempered distributions.

Before studying them it is convenient to see how far Definition 1 can be carried out.

Example 7. *Let $n \geq 1$ and $f(x) = \delta_0$, the delta function, i.e., the measure of mass one concentrated at the origin. Using (0.1) one finds that*

$$\widehat{\delta_0}(\xi) = \int_{\mathbb{R}^n} \delta_0(x)\, e^{-2\pi i(x\cdot\xi)} dx \equiv 1.$$

In fact, Definition 1 tells us that if μ is a bounded measure, then $\widehat{\mu}(\xi)$ represents a function in $L^\infty(\mathbb{R}^n)$.

Suppose that given $f(x) \equiv 1$ we want to find $\widehat{f}(\xi)$. In this case (0.1) cannot be used directly. It is necessary to introduce the notion of tempered distribution. For this purpose, we first need the following family of seminorms.

For each $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ we denote the seminorm $\|\!| \cdot \|\!|_{(\nu,\beta)}$ defined as

$$\|\!| f \|\!|_{(\nu,\beta)} = \|x^\nu \partial_x^\beta f\|_\infty = \sup_x |x^\nu \partial_x^\beta f(x)|.$$

We define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of the C^∞-functions decaying at infinity, i.e.,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{(\nu,\beta)} < \infty \quad \text{for any} \quad \nu, \beta \in (\mathbb{Z}^+)^n\}.$$

Observe that $C_0^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n)$ since $e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$.

No matter how large N is, $(1 + |x|^2)^{-N}$ is not in $\mathcal{S}(\mathbb{R}^n)$.

Not all members of $\mathcal{S}(\mathbb{R}^n)$ decay exponentially (Exercise).

The topology in $\mathcal{S}(\mathbb{R}^n)$ is given by the family of seminorms $\|\!| \cdot \|\!|_{(\nu,\beta)}$, $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$.

Definition 3. Let $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$. Then $\varphi_j \to 0$ as $j \to \infty$ if for any $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ one has that

$$\|\!|\varphi_j\|\!|_{(\nu,\beta)} \longrightarrow 0 \quad \text{as} \quad j \to \infty.$$

It is not difficult to show that φ_j converges a φ in $\mathcal{S}(\mathbb{R}^n)$ if and only if $d(\varphi_j, \varphi)$ tends to zero where d is the metric

$$d(\phi, \psi) = \sum_{\alpha,\beta\in\mathbb{N}^n} 2^{-(|\alpha|+|\beta|)} \frac{\|\phi - \psi\|_{\alpha,\beta}}{1 + \|\phi - \psi\|_{\alpha,\beta}}.$$

We observe that $\mathcal{S}(\mathbb{R}^n)$ endowed with the metric d is a complete metric space.

The relationship between the Fourier transform and the function space $\mathcal{S}(\mathbb{R}^n)$ is described in the formulae,

$$\begin{aligned} \partial_\xi^\alpha \widehat{f}(\xi) &= ((-2\pi i)^{|\alpha|} x^\alpha) f(x))^\wedge(\xi), \\ \widehat{(\partial_x^\beta f)}(\xi) &= (2\pi i)^{|\beta|} \xi^\beta \widehat{f}(\xi). \end{aligned} \tag{0.13}$$

More precisely, we have the following result (see Exercise 1.13).

Theorem 6. *The map* $\varphi \mapsto \widehat{\varphi}$ *is an isomorphism from* $\mathcal{S}(\mathbb{R}^n)$ *into itself.*

Thus $\mathcal{S}(\mathbb{R}^n)$ appears naturally associated to the Fourier transform. By duality we can define the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

Definition 4. We say that $\psi : \mathcal{S}(\mathbb{R}^n) \mapsto \mathbb{C}$ defines a *tempered distribution,* i.e., $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ if

1. Ψ is linear,
2. Ψ is continuous, i.e., if for any $\{\varphi_j\} \subseteq \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_j \to 0$ as $j \to \infty$, then the numerical sequence $\Psi(\varphi_j) \to 0$ as $j \to \infty$.

It is easy to check that any bounded function f defines a tempered distribution Ψ_f where

$$\Psi_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{0.14}$$

In fact, this identity allows us to see that any locally integrable function with polynomial growth at infinity defines a tempered distribution. In particular, we have the $L^p(\mathbb{R}^n)$ spaces with $1 \leq p \leq \infty$. The following example gives us a tempered distribution outside these function spaces.

Example 8. *In* $\mathcal{S}'(\mathbb{R})$ *define the principal value function of* $1/x$*, denoted by* $\mathrm{p.v.}\,\dfrac{1}{x}$*, by the expression*

$$\mathrm{p.v.}\,\frac{1}{x}(\varphi) = \lim_{\epsilon \downarrow 0} \int\limits_{\epsilon < |x| < 1/\epsilon} \frac{\varphi(x)}{x} dx,$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$*. Since* $1/x$ *is an odd function,*

$$\mathrm{p.v.}\frac{1}{x}(\varphi) = \int\limits_{|x|<1} \frac{\varphi(x) - \varphi(0)}{x} dx + \int\limits_{|x|>1} \frac{\varphi(x)}{x} dx. \qquad (0.15)$$

Therefore

$$|\mathrm{p.v.}\,\frac{1}{x}(\varphi)| \le 2\|\varphi'\|_\infty + 2\|x\varphi\|_\infty, \qquad (0.16)$$

and consequently $\mathrm{p.v.}\,\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$*.*

Now, given a $\Psi \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform can be defined in the following natural form.

Definition 5. Given $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ its *Fourier transform* $\widehat{\Psi} \in \mathcal{S}'(\mathbb{R}^n)$ is defined as

$$\widehat{\Psi}(\varphi) = \Psi(\widehat{\varphi}), \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{0.17}$$

Observe that for $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (0.7), (0.14), and (0.17) tell us that

$$\widehat{\Psi}_f(\varphi) = \Psi_f(\widehat{\varphi}) = \int_{\mathbb{R}^n} f(x)\widehat{\varphi}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(x)\varphi(x)dx = \Psi_{\widehat{f}}(\varphi).$$

Therefore, for $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ one has that $\widehat{\Psi}_f = \Psi_{\widehat{f}}$. Thus, Definition 5 is consistent with the theory of the Fourier transform developed in the previous sections.

Example 9. *Let* $f(x) \equiv 1 \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. *Using the previous notation, for any* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *it follows that*

$$\widehat{\Psi}_1(\varphi) = \Psi_1(\widehat{\varphi}) = \int_{\mathbb{R}^n} 1\,\widehat{\varphi}(x)dx = \varphi(0) = \int_{\mathbb{R}^n} \delta_0(x)\,\varphi(x)dx = \delta_0(\varphi).$$

Hence $\widehat{1} = \delta_0$. *We recall that in Example 7 we already saw that* $\widehat{\delta_0} = 1$.

Next we compute the Fourier transform of the tempered distribution in Example 8.

Example 10. *Combining Definition 5, Fubini's theorem, and the Lebesgue dominated convergence theorem we have that for any* $\varphi \in \mathcal{S}(\mathbb{R})$

$$\widehat{\textit{p.v.}\,\frac{1}{x}}(\varphi) = \textit{p.v.}\,\frac{1}{x}(\widehat{\varphi}) = \lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\widehat{\varphi}(x)}{x}dx$$

$$= \lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{1}{x}\Big(\int\limits_{-\infty}^{\infty} \varphi(y)e^{-2\pi i x y}\,dy\Big)dx$$

$$= \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \varphi(y) \Big(\int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx \Big) \, dy$$

$$= \int_{-\infty}^{\infty} \varphi(y) \Big(\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx \Big) dy$$

$$= -i\pi \int_{-\infty}^{\infty} \mathsf{sgn}(y) \; \varphi(y) dy,$$

where a change of variables and complex integration have been used to conclude that

$$\lim_{\epsilon \downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi i x y}}{x} dx = -2i \int\limits_0^\infty \frac{\sin(2\pi x y)}{x} dx$$

$$= -2i\, \mathsf{sgn}(y) \int\limits_0^\infty \frac{\sin(x)}{x} dx$$

$$= -i\pi\, \mathsf{sgn}(y).$$

This yields the identity

$$\widehat{\mathrm{p.v.}\, \frac{1}{x}}(\xi) = -\, i\pi\, \mathsf{sgn}(\xi).$$

The topology in $\mathcal{S}'(\mathbb{R}^n)$ can be described in the following form.

Definition 6. Let $\{\Psi_j\} \subset \mathcal{S}'(\mathbb{R}^n)$. Then $\Psi_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$ if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that $\Psi_j(\varphi) \longrightarrow 0$ as $j \to \infty$.

As a consequence of the Definitions 4, 6, we get the next extension of Theorem 6, whose proof we leave as an exercise.

Theorem 7. *The map* $\mathcal{F} : \Psi \mapsto \widehat{\Psi}$ *is an isomorphism from* $\mathcal{S}'(\mathbb{R}^n)$ *into itself.*

Combining the above results with an extension of Example 3 (see Exercise 1.2) we can justify the following computation related with the fundamental solution of the time dependent Schrödinger equation.

Example 11. $\widehat{e^{-4\pi^2 it|x|^2}} = \lim_{\epsilon\to 0^+} \widehat{e^{-4\pi^2(\epsilon+it)|x|^2}}$ *in* $\mathcal{S}'(\mathbb{R}^n)$*.*

From Exercise 1.2 it follows that

$$(\widehat{e^{-4\pi^2(\epsilon+it)|x|^2}})(\xi) = \frac{e^{-|\xi|^2/4(\epsilon+it)}}{[4\pi(\epsilon+it)]^{n/2}}.$$

Taking the limit $\epsilon \to 0^+$ *we obtain*

$$(\widehat{e^{-4\pi^2 it|x|^2}})(\xi) = \frac{e^{i|\xi|^2/4t}}{(4\pi it)^{n/2}}. \tag{0.18}$$

As an application of these ideas we introduce the Hilbert transform.

Definition 7. *For* $\varphi \in \mathcal{S}(\mathbb{R})$ *we define its Hilbert transform* $\mathcal{H}(\varphi)$ *by*

$$\mathcal{H}(\varphi)(y) \;=\; \frac{1}{\pi}\,\textit{p.v.}\frac{1}{x}\left(\varphi(y-\cdot)\right) \;=\; \frac{1}{\pi}\,\textit{p.v.}\,\frac{1}{x} * \varphi(y).$$

From (0.15), (0.16) it is clear that $\mathcal{H}(\varphi)(y)$ is defined for any $y \in \mathbb{R}$ and it is bounded by $g(y) = a|y| + b$, with $a, b > 0$ depending on φ. In particular we have that $\mathcal{H}(\varphi) \in \mathcal{S}'(\mathbb{R})$. Let us compute its Fourier transform.

Example 12. *From Example 10 and the identity*

$$\mathcal{H}(\varphi)(y) = \lim_{\epsilon\to 0} \left(\frac{1}{\pi}\frac{1}{x}\chi_{\{\epsilon<|x|<1/\epsilon\}} * \varphi\right)(y) \quad \textit{in} \quad \mathcal{S}'(\mathbb{R})$$

it follows that

$$\lim_{\epsilon\to 0} \widehat{\left(\frac{1}{\pi}\frac{1}{x}\chi_{\{\epsilon<|x|<1/\epsilon\}} * \varphi\right)}(\xi) = -i\,\mathrm{sgn}(\xi)\,\widehat{\varphi}(\xi).$$

This implies that

$$\widehat{\mathcal{H}(\varphi)}(\xi) = -i\,\textit{sgn}(\xi)\,\widehat{\varphi}(\xi), \quad \textit{for any} \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{0.19}$$

The identity (0.19) allows us to extend the Hilbert transform as an isometry in $L^2(\mathbb{R})$. *It is not hard to see that*

$$\|\mathcal{H}(\varphi)\|_2 = \|\varphi\|_2 \quad \textit{and} \quad \mathcal{H}(\mathcal{H}(\varphi)) = -\varphi.$$

Other properties of the Hilbert transform will be deduced in the exercises in Chapters 1 and 2 in [**?**].

In Definition 7 we have implicitly utilized the following result, which will be employed again in the applications at the end of this chapter.

Proposition 3. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\Psi \in \mathcal{S}'(\mathbb{R}^n)$. *Define*

$$\Psi * \varphi(x) = \Psi(\varphi(x - \cdot)). \tag{0.20}$$

Then

$$\Psi * \varphi \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$$

and

$$\widehat{\Psi * \varphi} = \widehat{\Psi}\widehat{\varphi}, \tag{0.21}$$

where $\widehat{\Psi}\widehat{\varphi} \in \mathcal{S}'(\mathbb{R}^n)$ *is defined as* $\widehat{\Psi}\widehat{\varphi}(\phi) = \widehat{\Psi}(\widehat{\varphi}\phi)$ *for any* $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. It is left as an exercise. $\square$ $\square$