Teoria Espectral — Homework 9

Due on 25/05/2021

(1) Let A be a bounded linear operator in a Hilbert space $\mathcal{H}$. Show that
$$\|A^*A\| = \|A\|^2.$$

(2) The objetive of this problem is to prove the Lax-Milgram theorem in Hilbert spaces. Let $\mathcal{H}$ be a complex Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a sesquilinear form (which means: B is linear in both entries).

(a) Show that B is continuous if and only if there exists $M \geq 0$ such that
$$|B(x,y)| \leq M\|x\|\|y\| \quad \forall x, y \in \mathcal{H}.$$
Hint: For the $\Longrightarrow$ part use the Banach-Steinhaus theorem.

(b) Show that B is continuous if and only if if there exists $A \in \mathcal{B}(\mathcal{H})$ such that
$$B(x,y) = \langle x, Ay\rangle \qquad \forall x, y \in \mathcal{H}.$$
A sesquilinear form B is called **coercive** if $B(x,x) \geq c\,\|x\|^2$ for some $c > 0$.

(c) Prove the Lax-Milgram Theorem. If B is continuous and coercive on $\mathcal{H}$ then given $w \in \mathcal{H}$ there exists a unique element $x \in \mathcal{H}$ such that $B(u,x) = \langle u, w\rangle$ for all $u \in \mathcal{H}$. For such x one has $\|x\| \leq \frac{1}{c}\|w\|$ where $c > 0$ is the bound from below of the form (that is, $B(u,u) \geq c\,\|u\|^2$ for all $u \in \mathcal{H}$).

(3) Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Show that the spectrum of A is contained in $[n, M]$ with $m = \inf \dfrac{\langle Au, u\rangle}{\|u\|^2}$. Moreover m and M belong to the spectrum of T.

Hint: Use the Lax-Milgram theorem.