Teoria Espectral Homework 8
Due on 11/05/2021

(1) If A is a closed operator and B is A-bounded operator, then
 (i) $\mathcal{H} = (\mathcal{D}(A), [\cdot,\cdot])$ is a Hilbert space with inner product
$$[\phi, \psi] = (\phi, \psi) + (A\phi, A\psi).$$
 (ii) $B \in \mathcal{B}(\mathcal{H}, H)$.

(2) If B is closed and $\rho(A) \neq \emptyset$. Prove that the following affirmations are equivalent:
 (i) B is A-bounded.
 (ii) $B\,(A - z)^{-1} \in \mathcal{B}(H)$ for some $z \in \rho(A)$.
 (iii) $B\,(A - z)^{-1} \in \mathcal{B}(H)$ for all $z \in \rho(A)$.

(3) Let $H_0 = -\Delta : H^2(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the free Hamiltonian and let $z \in \rho(H_0) = \mathbb{C} \setminus [0, +\infty)$.

 (i) Prove that
$$R_0(z)g := (H_0 - z)^{-1} f = \mathcal{R}_z * g, \ \forall g \in L^2(\mathbb{R}^n)$$
 where $\mathcal{R}_z = \left((|\xi|^2 - z)^{-1}\right)^{\vee}$. Check that $R_0(z) \in \mathcal{B}(L^2(\mathbb{R}^n))$.
 (ii) In case $n = 1$, prove that
$$\mathcal{R}_z(x) = \frac{e^{i\sqrt{z}|x|}}{2\sqrt{z}}, \quad \text{where } \mathrm{Im}\sqrt{z} > 0.$$
 Hint: Use the Residue Theorem.

 (iii) If $z = \lambda + i\eta$ with $\lambda \geq 0$, prove that $\lim\limits_{\eta \to 0} R_0(\lambda + i\eta)$ does not exist $\mathcal{B}(L^2(\mathbb{R}^n))$.