(1) (Spectrum of the multiplication operator). Let $q : \mathbb{R} \to \mathbb{R}$ be such that q is mensurable and finite *a.e.* Define M_q the multiplication operator by q:

$$\begin{cases} D(M_q) = \{\phi \in L^2(\mathbb{R}) \ : \ q\,\phi \in L^2(\mathbb{R})\}, \\ M_q\phi = q\,\phi. \end{cases}$$

(i) Prove that M_q is a closed operator densely defined.
(ii) Prove that M_q is a sefl-adjoint operator (*i.e.* $M_q = M_q^*$).
Remark: In particular, this implies that $\sigma(M_q) \subset \mathbb{R}$.
(iii) Let $r \in \mathbb{R}$, define $I_r = \{x \in \mathbb{R} \ : \ q(x) = r\}$. Prove that

$$\mathrm{av}(M_q) = \{r \in \mathbb{R} \ : \ \lambda(I_r) > 0\},$$

where λ denotes the Lebesgue measure on $\mathbb{R}$.
(iv) Suposse q continuous, show that $\sigma(M_q) = \overline{q(\mathbb{R})}$.
Hint: Prove that $q(\mathbb{R}) \subset \sigma(M_q)$, suppose that $r = q(t) \in \rho(M_q)$ and use the function $\phi_\epsilon(x) = \frac{\chi_{B_\epsilon}}{\sqrt{\lambda(B_\epsilon)}}$, where $B_\epsilon = \{x \in \mathbb{R} \ : \ |q(x) - r| < \epsilon\}$, to obtain a contradiction.
(v) Coming back to the general case (*i.e.* $q \in L^1_{\mathrm{loc}}(\mathbb{R})$), define the **essential image** of q by

$$\mathrm{Im}_e(q) = \{r \in \mathbb{R} : \lambda(q^{-1}(B(r,\epsilon))) > 0, \ \forall\, \epsilon > 0\},$$

where $B(r,\epsilon) = \{y \in \mathbb{R} \ : \ |y - r| < \epsilon\}$. Prove that $\mathrm{Im}_e(q) \subset \overline{q(\mathbb{R})}$, that equality holds if q is continuous, but it does not hold in general.
(vi) Prove that $\sigma(M_q) = \mathrm{Im}_e(q)$.

(2) Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ be a closed linear operator such that $A \subseteq A^\star$.
(i) Let $z = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$ (*i.e.* $\beta \neq 0$). Prove that $\mathrm{Im}(A - z)$ is a closed set.
(ii) Let $z = \alpha + i\beta \in \mathbb{C}\setminus\mathbb{R}$ and $\eta \in \mathbb{C}$ such that $|\eta| < |\beta|$. Prove that $N(A^* - (z+\eta)) \cap N(A^* - z)^\perp = \{0\}$.
(iii) Let M and N be two subspaces of H satisfying $\dim M > \dim N$. Show that $M \cap N^\perp \neq \{0\}$.
(iv) Deduce that $z \mapsto \dim N(A^* - z)$ is constant on the upper half-plane $\mathbb{C}_+$, where $\mathbb{C}_+ = \{z \in \mathbb{C} : Im\, z > 0\}$.

(3) Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ be a linear operator such that $A = A^\star$ and $M \in \mathbb{R}$. Prove that

$$A \geq M \quad \Longleftrightarrow \quad \sigma(A) \subset [M, +\infty).$$

Hint: If $\xi \in \rho(A)$, show that the spectral ray $r(\xi)$ of $(A - \xi)^{-1}$ is given by $r(\xi) = \dfrac{1}{d(\xi, \sigma(A))}$.