Teoria Espectral Homework 6

Due on 27/04/2021

We denote by X and Y Banach spaces.

(1) Let $\mathcal{M}$ be a subspace of $X \times Y$. Prove that $\mathcal{M}$ is the graph of a linear operator if and only if $\mathcal{M}$ does not contain points of the form $(0, v)$, $v \neq 0$.

(2) Let $T : D(T) \subset X \to Y$ be a linear operator. Prove that if T is a closable operator then

$$D(\overline{T}) = \{u \in X \ : \ u_j(\in D(T)) \overset{X}{\to} u \ \text{and} \ \{Tu_j\} \ \text{is a Cauchy sequence in } Y\}.$$

(3) Consider the operators A_j, $j = 0, 1, 2$, defined by

$$\begin{aligned} D(A_0) &= H^1([-\pi, \pi]), \\ D(A_1) &= \{\phi \in \mathcal{D}(A_0) \ /\phi(-\pi) = \phi(\pi)\}, \\ D(A_2) &= \{\phi \in \mathcal{D}(A_1) \ / \ \phi(-\pi) = \phi(\pi) = 0\}, \end{aligned}$$

and

$$A_j = \frac{1}{i}\frac{d}{dx}, \ j = 0, 1, 2.$$

(i) Prove that A_j is closed for $j = 0, 1, 2$.
(ii) Show that $\sigma(A_0) = \sigma(A_2) = \mathbb{C}$ and $\sigma(A_1) = \mathbb{Z}$.

(4) (Operator with empty spectrum) We Define $A^{\pm}$ by

$$\begin{aligned} D(A^{\pm}) &= \{\phi \in D(A_0) \ : \ \phi(\pm\pi) = 0\}, \\ A^{\pm}\phi &= A_0\phi = \frac{1}{i}\,\phi'. \end{aligned}$$

Show that $\sigma(A^{\pm}) = \emptyset$.

(5) (An operator without eigenvalues)
Let $D(M) = L^2([-\pi, \pi]) = L^2_{\text{per}}$.

$$\begin{aligned} M : D(M) &\to L^2_{\text{per}} \\ f &\mapsto Mf(x) = x\,f(x) \qquad a.e. \ \ x \in [-\pi, \pi]. \end{aligned}$$

Prove that
(i) $M \in \mathcal{B}(L^2([-\pi, \pi]))$;
(ii) $M\phi = \lambda\,\phi \implies \phi = 0$;
(iii) $\sigma(M) = [-\pi, \pi]$.