Due on 15/04/2021.

(1) For a given $f \in L^2(\mathbb{R}^n)$ prove that the following statements are equivalent:
   (i) $g \in L^2(\mathbb{R}^n)$ is the partial derivative of $f \in L^2(\mathbb{R}^n)$ with respect to the kth variable according to Definition 1.2 (see notes Fourier Transform).
   (ii) There exists $g \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)\partial_{x_k}\phi(x)\,dx = -\int_{\mathbb{R}^n} g(x)\phi(x)\,dx \tag{0.1}$$

   for any $\phi \in C_0^\infty(\mathbb{R}^n)$. In general, if (0.1) holds for two distributions f, g, then one says that g is the kth partial derivative of f in the distribution sense.
   (iii) There exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\|f_j - f\|_2 \to 0 \quad \text{as} \quad j \to \infty,$$

   and $\{\partial_{x_k} f_j\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.
   (iv) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$.
   (v)

$$\sup_{h>0} \int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} \right|^2 dx < \infty.$$

(2) Let $\phi(x) = e^{-|x|}$, $x \in \mathbb{R}$.
   (i) Prove that

$$\phi(x) - \phi''(x) = 2\delta, \tag{0.2}$$

   (a) in the distribution sense, i.e., $\forall \varphi \in C_0^\infty(\mathbb{R})$,

$$\int \phi(x)(\varphi(x) - \varphi''(x))\,dx = 2\varphi(0);$$

   (b) by taking the Fourier transform in (0.2).

   (ii) Prove that given $g \in L^2(\mathbb{R})$ (or $H^s(\mathbb{R})$) the equation

$$\Big(1 - \frac{d^2}{dx^2}\Big) f = g$$

   has solution $f = \frac{1}{2}\, e^{-|\cdot|} * g \in H^2(\mathbb{R})$ (or $H^{s+2}(\mathbb{R})$).

(3) (a) Let $u(x)$ be the function which is equal to $\ln(x)$ for $x > 0$ and 0 for $x \le 0$. Then u is locally integrable. Compute the distribution derivative $\frac{du}{dx}$.
   (b) Compute the distribution derivative of the function $\ln(|x|) \in L^1_{\text{loc}}(\mathbb{R})$.

(4) Define $f \in L^\infty(\mathbb{R}^2)$ by $f(x_1, x_2) = \dfrac{x_1 x_2}{x_1^2 + x_2^2}$ for $(x_1, x_2) \ne 0$. Compute $\partial_{x_i} f$ and $\partial_{x_1 x_2} f$.