Due on 08/04/2021.

(1) Prove the following formula in $\mathcal{S}'(\mathbb{R}^n)$:

$$(\widehat{e^{-a|x|^2}})(\xi) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}, \quad \mathcal{R}e\, a \geq 0, \quad a \neq 0,$$

where $\sqrt{a}$ is defined in the branch $\mathcal{R}e\, a > 0$.

(2) Show that if $f \in L^1(\mathbb{R}^n)$, $f \not\equiv 0$, with compact support, then for any $\epsilon > 0$, $\widehat{f} \notin L^1(e^{\epsilon|x|}dx)$.

(3) Consider the initial value problem associated to the wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = g(x), \end{cases} \tag{0.1}$$

Prove that:

(i) If $f, g \in C_0^\infty(\mathbb{R}^n)$ are real valued functions, then the solution can be described by the following expression:

$$w(x,t) = U'(t)f + U(t)g = \cos(Dt)f + \frac{\sin(Dt)}{D}g, \tag{0.2}$$

where $\widehat{U(t)h}(\xi) = \dfrac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\widehat{h}(\xi)$ and $\widehat{Dh}(\xi) = 2\pi|\xi|\,\widehat{h}(\xi)$.

(ii) If f, g are supported in $\{x \in \mathbb{R}^3 : |x| \leq M\}$ show that $w(\cdot, t)$ is supported in $\{x \in \mathbb{R}^3 : |x| \leq M + t\}$.

(iii) Suppose $n = 3$ and $f \equiv 0$, prove that

$$w(x,t) = \frac{1}{4\pi t} \int_{\{|y|=t\}} g(x+y)\, dS_y.$$

Hint: Deduce and apply the identity

$$\int_{\{|x|=t\}} e^{2\pi i \xi \cdot x}\, dS_x = 4\pi t\, \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}.$$

If $g \in C_0^\infty(\mathbb{R}^3)$ is supported in $\{x \in \mathbb{R}^3 : |x| \leq M\}$, where is the support of $w(\cdot, t)$?

(iv) If $f \in H^2(\mathbb{R}^3)$ and $g \in H^1(\mathbb{R}^3)$ show that $w \in C([0,T] : H^2(\mathbb{R}^3))$.