(1) (i) A function $f \in \mathcal{S}(\mathbb{R}^n)$ is called homogeneous of degree de a, if

$$f(\lambda x) = \lambda^a f(x), \quad \forall \lambda > 0, \ \forall x \in \mathbb{R}^n.$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$, define $\phi_\lambda(x) := \lambda^{-n}\phi(\lambda^{-1}x)$, if λ is positive. Prove that

$$\int_{\mathbb{R}^n} f(x)\phi_\lambda(x)dx = \lambda^a \int_{\mathbb{R}^n} f(x)\phi(x)dx, \ \forall \lambda > 0.$$

(ii) Let $T \in \mathcal{S}'(\mathbb{R}^n)$, we say that T is homogeneous of degree a if

$$\langle T, \phi_\lambda \rangle = \lambda^a \langle T, \phi \rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Prove that if $T \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree a, then $\widehat{T}$ is homogeneous of degree $-n-a$.

(iii) Let $\frac{n}{2} < a < n$, define $f(x) = |x|^{-a}$. Prove that

$$f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) \subseteq \mathcal{S}'(\mathbb{R}^n).$$

Use (ii) to show that there exists a constant $c_{a,n}$ such that

$$\widehat{f}(\xi) = c_{a,n}|\xi|^{a-n}.$$

(2) (i) Define

$$\text{v.p.}(\frac{1}{x}) : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \quad \phi \mapsto \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\phi(x)}{x} dx.$$

Prove that $\text{v.p.}(\frac{1}{x}) \in \mathcal{S}'(\mathbb{R})$ and

$$\left(\text{v.p.}(\frac{1}{x})\right)^{\wedge}(\xi) = -i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \operatorname{sgn}(\xi).$$

(ii) Define

$$(x \pm i0)^{-1} : \mathcal{S}(\mathbb{R}) \to \mathbb{C}, \quad \phi \mapsto \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{\phi(x)}{x \pm i\epsilon} dx.$$

Prove that $(x \pm i0)^{-1} \in \mathcal{S}'(\mathbb{R})$ and

$$(x \pm i0)^{-1} = \text{v.p.}(\frac{1}{x}) \mp i\pi\delta, \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Find the Fourier transform of $(x \pm i0)^{-1}$.

(3) (Characterization of the space $\mathcal{S}'(\mathbb{R}^n)$) We say that a linear functional

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}, \quad \phi \mapsto \langle T, \phi \rangle$$

is continuous if and only if

$$\phi_n \xrightarrow{d} \phi \quad \Rightarrow \quad \langle T, \phi_n \rangle \to \langle T, \phi \rangle, \quad \forall \ (\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n), \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

We define the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ as

$$\mathcal{S}'(\mathbb{R}^n) := \{T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} \ : \ T \text{ linear and continuous}\}.$$

Notice that $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$. Prove that $T \in \mathcal{S}'(\mathbb{R}^n)$ if and only if there exist $C > 0$ and $k \in \mathbb{N}$ such that

$$|\langle T, \phi \rangle| \leq C \sum_{|\alpha|,|\beta| \leq k} \|\phi\|_{\alpha,\beta}, \ \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$