Teoria Espectral
Due on 25/03/2021

(1) Let $n \geq 1$ and $f(x) = e^{-2\pi|x|}$. Show that

$$\hat{f}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{1}{(1+|\xi|^2)^{(n+1)/2}}.$$

(2) (i) Prove that if $f \in L^1(\mathbb{R}^n)$ e $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi)$.

(ii) If $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$ with $1/p + 1/p' = 1$, $1 < p < \infty$, prove that $f * g \in C_\infty(\mathbb{R}^n)$. What can we say in the case $p = 1, \infty$?

(iii) If $f \in L^1(\mathbb{R}^n)$, with f continuous at the origin and $\widehat{f} \geq 0$, prove that $\widehat{f} \in L^1(\mathbb{R}^n)$.

(3) Let $B_R := \{x \in \mathbb{R}^n \ : \ |x| < R\}$ and χ_{B_R} a characteristic function of the set B_R. Define

$$S_R : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \ f \longmapsto (\chi_{B_R}\widehat{f})^{\vee}.$$

(i) Prove that $S_R \in \mathcal{B}(L^2(\mathbb{R}^n))$ and that $\|S_R\|_{\mathcal{B}(L^2)} \leq 1$.

(ii) Show that $\forall\ f \in L^2(\mathbb{R}^n)$, $S_R f \underset{R\to+\infty}{\longrightarrow} f$ in $L^2(\mathbb{R}^n)$.

(iii) Deduce that for any $f \in L^2(\mathbb{R}^n)$, there is a sequence $R_n \underset{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{B_{R_n}} \widehat{f}(\xi) e^{ix.\xi} d\xi \underset{n\to+\infty}{\longrightarrow} f(x), \quad \text{q.t.p. } x \in \mathbb{R}^n.$$

(iv) Prove that for any $f \in L^2(\mathbb{R}^n)$,

$$: \xi \mapsto \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{B_R} f(x) e^{-ix.\xi} dx \underset{R\to+\infty}{\longrightarrow} \widehat{f}, \quad \text{em } L^2(\mathbb{R}^n).$$

(4) (Topology on $\mathcal{S}(\mathbb{R}^n)$) Define the map

$$\begin{aligned} d : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathbb{R}_+, \\ (\phi, \psi) &\longmapsto \sum_{\alpha,\beta\in\mathbb{N}^n} 2^{-(|\alpha|+|\beta|)} \frac{\|\phi-\psi\|_{\alpha,\beta}}{1+\|\phi-\psi\|_{\alpha,\beta}}. \end{aligned}$$

Prove that $(\mathcal{S}(\mathbb{R}^n), d)$ is a complete metric space and for any sequence $(\phi_n)_n \subset \mathcal{S}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$, it holds

$$\phi_n \overset{d}{\to} \phi \quad \Longleftrightarrow \quad \|\phi_n - \phi\|_{\alpha,\beta} \to 0, \ \forall \alpha, \ \beta \in \mathbb{N}^n.$$

Finally, show that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad \phi \mapsto \widehat{\phi},$$

is a topological isomorphism.