Teoria Espectral Homework 1

Due on 18/03/2021

(1) (Volterra's operator) Let $a,\ b \in \mathbb{R}$ with $a < b$, consider the Volterra operator

$$\begin{array}{rcl} V : C([a,b];\mathbb{C}) & \longrightarrow & C([a,b];\mathbb{C}), \\ f & \longmapsto & Vf : t \mapsto \int_a^t f(s)ds. \end{array}$$

(i) Prove that $V \in \mathcal{B}(C([a,b];\mathbb{C}))$, where the space $C([a,b];\mathbb{C})$ is equipped with the norm $\|\cdot\|_{L^\infty}$ defined by $\|f\|_{L^\infty} := \sup_{t\in[a,b]} |f(t)|$.

(ii) Prove that $\forall n \in \mathbb{N}$, $n \geq 1$, it holds that

$$V^n f(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)ds.$$

(iii) Show that

$$\|V^n\|_{L^\infty} \leq \frac{1}{n!}(b-a)^n, \quad \forall n \in \mathbb{N}.$$

Conclude that the unique fixed point of V is $f = 0$.

(2) The purpose of this exercise is to show that the Fourier transform

$$\mathcal{F} : L^1(\mathbb{R}) \to C^0_\infty(\mathbb{R}), \quad f \mapsto \widehat{f},$$

is not surjective. We recall that

$$C^0_\infty(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \ : \ f \text{ is continuous and } \lim_{|x|\to\infty} f(x) = 0\}.$$

(i) Prove that $(C^0_\infty(\mathbb{R}), \|\cdot\|_{L^\infty})$ is a Banach space, where $\|f\|_{L^\infty} := \sup_{x\in\mathbb{R}} |f(x)|$.

(ii) Use the Fourier inversion formula to show that $\mathcal{F}$ is injective.

(iii) Suppose that $\mathcal{F}$ is surjective, use the open map theorem to deduce that there exists a constant $c > 0$ such that

$$\|f\|_{L^1} \leq c\|\widehat{f}\|_{L^\infty}, \quad \forall\ f \in L^1(\mathbb{R}). \tag{0.1}$$

(iv) Let $A \geq 1$, define

$$\phi_A := \chi_{[-A,A]}, \quad \psi_A := \phi_A * \phi_1 \quad \text{e} \quad g_A := \widehat{\psi_A}.$$

Prove that

$$\|\widehat{g_A}\|_{L^\infty} = 1, \quad g_A(x) = 2\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin(Ax)\sin(x)}{x^2}, \quad \|g_A\|_{L^1} \underset{A\to+\infty}{\longrightarrow} +\infty.$$

Use this to establish a contradiction in (0.1).