Teoria Espectral Homework 12
Due 22/06/2020

(1) Let $A : D \to X$ be a closed operator in a Banach space X and $f \in C([0,T]; D)$. Let $u(t) = \int_0^t f(s)\,ds$. Prove that $u \in C([0,T]; D)$ and $Au(t) = \int_0^t Af(s)\,ds$.

(2) Define for $t > 0$

$$(S(t)g)(x) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy, \quad x \in \mathbb{R}^n$$

where $g : \mathbb{R}^n \to \mathbb{R}$ and Φ is the fundamental solution of the heat equation. Set $S(0)g = g$.

(a) Prove that $\{S(t)\}_{t\geq 0}$ is a semigroup of contractions in $L^2(\mathbb{R}^n)$.
(b) Show that $\{S(t)\}_{t\geq 0}$ is not a semigroup of contractions in $L^\infty(\mathbb{R}^n)$.

(3) Let $\{S(t)\}_{t\geq 0}$ be a semigroup of contractions in X, with infinitesimal generator A. Inductively define $D(A^k) = \{x \in D(A^{k-1}) : \; A^{k-1}x \in D(A)\}$, $k = 2, \ldots$. Show that if $x \in D(A^k)$, for some k, then $S(t)x \in D(A^k)$ for all $t \geq 0$.

(4) Use the previous exercise to prove that if u is a solution in $X = L^2(U)$ of

$$\begin{cases} \partial_t u - \Delta u = 0 \quad \text{em } \; U_T \\ u = 0 \quad \text{em } \; \partial U \times [0,T] \\ u = g \quad \text{em } \; U \times \{t = 0\}, \end{cases}$$

with $g \in C_c^\infty(U)$, then $u(\cdot, t) \in C^\infty(U)$ for each $0 \leq t \leq T$.

(5) Consider the initial value problem associated to the wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0, \quad x \in \mathbb{R}^n, \;\; t \in \mathbb{R}^+, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = g(x), \end{cases} \tag{0.1}$$

We already proved that for $f, g \in C_0^\infty(\mathbb{R}^n)$ real valued functions, then the solution can be described by the following expression:

$$w(x,t) = U'(t)f + U(t)g = \cos(Dt)f + \frac{\sin(Dt)}{D}g, \tag{0.2}$$

where $\widehat{U(t)h}(\xi) = \dfrac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\widehat{h}(\xi)$ and $\widehat{Dh}(\xi) = 2\pi|\xi|\,\widehat{h}(\xi)$.

(a) Are $\{U(t)\}_{t\geq 0}$ and $\{U'(t)\}_{t\geq 0}$ strongly continuous (one-parameter) unitary groups in $H^s(\mathbb{R}^n)$?
(b) Are $\{U(t)\}_{t\geq 0}$ and $\{U'(t)\}_{t\geq 0}$ C_0 semigroups of operators in $H^s(\mathbb{R}^n)$?