Teoria Espectral Homework 11
Due 15/06/2021

(1) Consider the linear initial value problem

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t), & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{0.1}$$

(i) Show that solutions of (0.1) form a strongly continuous one-parameter unitary group in $L^2(\mathbb{R}^n)$.
(ii) Prove that the infinitesimal generator operator A in (0.1) is the Laplacian Δ with $D(A) = H^2(\mathbb{R}^n)$.

(2) Let χ be a mensurable real-valued function on $\mathbb{R}$. We define $U(t)$ by

$$(U(t)\psi)(x) = e^{it\chi(x)} \cdot \psi(x). \tag{0.2}$$

$U(t)$ is called **local gauge transformation**.
(i) Show that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group in a Hilbert space $\mathcal{H}$.
(ii) Prove that the operator A defined by

$$D(A) = \left\{\psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |\chi(x)\psi(x)|^2\, dx < \infty\right\}$$

and

$$A\psi = T_\chi \psi = \chi(\cdot)\varphi(\cdot)$$

is the infinitesimal generator of the local gauge transformation (0.2).

(3) Let A be the infinitesimal generator of a C_0 semigroup of contractions $\{S(t)\}_{t\geq 0}$. If

$$A_\lambda = \lambda A R_\lambda(A) = \lambda^2 R_\lambda(A) - \lambda\, I,$$

prove that

$$S(t)x = \lim_{\lambda\to\infty} e^{A_\lambda} x \quad \text{for } x \in X.$$

(4) Let A be the infinitesimal generator of a C_0 semigroup of contractions $\{S(t)\}_{t\geq 0}$. Show that the resolvent set of A contains the open positive half-plane, i.e. $\rho(A) \supseteq \{\lambda : \operatorname{Re}\lambda > 0\}$ and for such λ,

$$\|R_\lambda(A)\| \leq \frac{1}{\operatorname{Re}\lambda}.$$

(5) Show that a linear operator A is the infinitesimal generator of a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ if and only if
(i) A is closed and $\overline{D(A)} = X$,
(ii) The resolvent set satisfies that $\{\lambda : \operatorname{Im}\lambda = 0, \lambda > \omega\} \subset \rho(A)$ and for such λ

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda - \omega}.$$