Teoria Espectral Homework 10

Due 1/06/2021

(1) Show that a linear combination of functions

$$\begin{cases} \mathbb{R} \to \mathbb{C} \\ x \mapsto \dfrac{1}{x-\lambda} \end{cases}$$

where $\lambda \in \mathbb{C}\backslash\mathbb{R}$, is dense in the space $C_0(\mathbb{R})$ of continuous functions in $\mathbb{R}$ with limit 0 at $\pm\infty$ with the *sup* norm.

Hint: Use Stone-Weierstrass theorem.

(2) Show that $E_0(\lambda) = \mathcal{F}^{-1}\xi_{\{|\xi|^2 \leq \lambda\}}\mathcal{F}$ is the spectral family associated to H_0.

(3) Let (M, μ) be a measurable space with measure μ finite. Suppose that f is measurable, real in M which is finite a.e.$[\mu]$. Prove that the operator

$$\varphi \overset{T_f}{\mapsto} f\varphi \quad \text{in} \quad L^2(M, d\mu)$$

with domain

$$D(T_f) = \{\varphi | \ f\varphi \in L^2(M, d\mu)\}$$

is self-adjoint and $\sigma(T_f)$ is the essential image of f.

(4) Let $\mathcal{H}$ be a Hilbert space and $T : D(T) \subset \mathcal{H} \to \mathcal{H}$. Show that there exists a unique map

$$\begin{cases} L^\infty(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) \\ f \mapsto f(T) \end{cases}$$

continuous with norm ≤ 1, with the following properties:

(i) The map is a ring homomorphism;
(ii) $f(T)^* = \bar{f}(T)$ for all f;
(iii) $f(T) > 0$ if $f > 0$;
(iv) If f_n converges pointwise to $f \in L^\infty(\mathbb{R})$ and $\|f_n\|$ is bounded, then $f_n(T) \to f(T)$ strongly;
(v) If $f_n \in L^\infty$ converges pointwise to the identity function $x \mapsto x$, and $|f_n(x)| \leq |x|$, we have

$$f_n(T)v \to Tv$$

for all $v \in D(T)$.

We recall de definition of ring homomorphism:

Let R and S be rings. Then $\phi : R \to S$ is a ring homomorphism if
(a) *ϕ is homomorphism of additive groups: $\phi(a + b) = \phi(a) + \phi(b)$ and*
(b) *ϕ preserves multiplication: $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$.*