Teoria Espectral

1. Unbounded Operators

These notes are intend to introduce the unbounded operators and several notions and properties related to them. The notes are sketchy and you might consult some additional textbooks.

- M. Reed and B. Simon, Methods of Modern Mathematical Physics, Volumes 1, 2
- E. Hille, Methods in Classical and Functional Analysis
- T. Kato, Perturbation Theory

We will use the following notation. We will denote X, Y to be Banach spaces. We will use $B(z,R)$ to denote an open ball with center z and radius R.

1.1. **Closed operators.**

Definition 1.1. *A linear operator $T : D(T) \subset X \to Y$ is* **closed** *if and only if for all sequence $\{\phi_n\} \subset D(T)$ such that*

$$\phi_n \xrightarrow{X} \phi \quad \text{and} \quad T\phi_n \xrightarrow{Y} \psi$$

then

$$\phi \in D(T) \quad \text{and} \quad T\phi = \psi,$$

if and only if the graph

$$G(T) = \{(\phi, T\phi) \ : \ \phi \in D(T)\}$$

is a closed set in $X \times Y$.

Remark 1.2. *A linear closed operator is the best we can have after a linear continuous operator.*

Example 1.3. *The operator H_0 defined by*

$$\begin{cases} D(H_0) = H^2(\mathbb{R}^n) \\ H_0 f = -\Delta f \end{cases}$$

is a closed operator.

It is not difficult to show that $H_0 = \mathcal{F}^{-1} M_0 \mathcal{F}$ where

$$\begin{cases} D(M_0) = \{\phi \in L^2(\mathbb{R}^n) \ : \ |\xi|^2\phi \in L^2(\mathbb{R}^n)\} \\ M_0\phi = |\xi|^2\phi. \end{cases}$$

Affirmation: *M_0 is closed.*

Indeed, let $\{\phi_n\} \subset D(M_0)$ *such that* $\phi_n \to \phi$ *in* L^2 *and* $M_0\phi_n \to \psi$ *in* L^2. *Then there exists a subsequence* $\{\phi_{n_k}\}$ *of* $\{\phi_n\}$ *such that*

$$\begin{cases} \phi_{n_k}(x) \to \phi(x) \\ |x|^2\phi_{n_k}(x) \to \psi(x) \end{cases} \textit{almost every} \quad x \in \mathbb{R}^n.$$

This implies that $|x|^2\phi(x) = \psi(x)$ *a.e. Hence* $|\cdot|^2\phi \in L^2(\mathbb{R}^n)$. *Thus* $\phi \in \mathcal{D}(M_0)$ *and* $\psi = M_0\phi$. *It follows that* H_0 *is closed.*

Exercise 1.4. *If* $A : D(A) \subset X \to Y$ *is bounded, show that*

$$A \quad \textit{is closed} \quad \Longleftrightarrow \quad D(A) \quad \textit{is closed in } X.$$

Exercise 1.5. *Let*

$$\begin{cases} T : D(T) \subset X \to Y & \textit{be a closed operator,} \\ A : D(A) \subset X \to Y & \textit{be a bounded operator and} \quad D(T) \subset D(A). \end{cases}$$

Show that $T + A : D(T) \subset X \to Y$ *is a closed operator and*

$$(T + A)\phi = T\phi + A\phi.$$

Remark 1.6. *The perturbation of a closed operator by a bounded operator is a closed operator.*

Definition 1.7. *Let* $T : D(T) \subset X \to Y$ *and* $S : D(S) \subset X \to Y$ *be linear operators. The sum of* T *and* S *is given by*

$$\begin{cases} D(T + S) = D(T) \cap D(S) \\ (T + S)\phi = T\phi + S\phi \quad \forall \phi \in D(T + S). \end{cases}$$

Definition 1.8.

(1) *Let* $T : D(T) \subset X \to Y$ *be a linear operator. The* **kernel** *of the operator* T *is defined by*

$$N(T) = \ker T = \{\phi \in D(T) : \; T\phi = 0\} \textit{ which a subspace of } \; D(T).$$

The **image** *of the operator* T *is defined by*

$$\mathrm{Im}(T) = R(T) = \{T\phi \; : \; \phi \in D(T)\} \textit{ which a subspace of } \; Y.$$

(2) *Let* $T : D(T) \subset X \to Y$ *be an injective linear operator, we define* T^{-1} *by*

$$\begin{cases} D(T^{-1}) = R(T) \\ T^{-1}T\phi = \phi, \quad \forall \phi \in D(T). \end{cases}$$

Thus $T^{-1} : R(T) \subset Y \to X$.

(3) *If* $T : D(T) \subset X \to Y$, $S : D(S) \subset Y \to Z$ *are two linear operators, we define* $S \circ T$ *by*

$$\begin{cases} D(S \circ T) = \{\phi \in D(T) \ : \ T\phi \in D(S)\} \\ S \circ T(\phi) = S(T\phi). \end{cases}$$

Some remarks on the graph of a linear operator $T : D(T) \subset X \to Y$.

(1) T is closed $\iff$ $G(T)$ is closed.

(2) $G(T)$ closed $\not\Rightarrow$ $D(T)$ is closed.

Example 1.9. H_0 *is a closed linear operator but* $D(H_0) = H^2(\mathbb{R}^n)$ *is not closed in* $L^2(\mathbb{R}^n)$. *Since* $\overline{H^2(\mathbb{R}^n)} = L^2(\mathbb{R}^n)$ *this would imply that* $H^2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ *which is false.*

Theorem 1.10 (Closed Graph Theorem)**.** *Let* X, Y *be Banach spaces. If* $T : X \to Y$ *is a closed linear operator, then* $T \in \mathcal{B}(X, Y)$.

Remark 1.11. *Note that the operator* T *is required to be everywhere-defined, i.e., the domain* $D(T)$ *of* T *is* X.

Example 1.12. *If* $T : D(T) \subset X \to Y$ *is a closed operator and* $S : X \to X$ *is a bounded operator.* $R(S) = \operatorname{Im} S \subset D(T)$. *Then* $T \circ S \in \mathcal{B}(X, Y)$.

$T \circ S$ *is closed. Let* $\{\phi_n\} \subset X = D(T \circ S)$ *such that*

$$\begin{cases} \phi_n \xrightarrow{X} \phi \\ (T \circ S)\phi_n \xrightarrow{Y} \psi \end{cases}$$

Since S *is continuous we have that*

$$\begin{cases} S\phi_n \xrightarrow{X} S\phi \\ T(S\phi_n) \xrightarrow{Y} \psi. \end{cases}$$

On the other hand, since T *is closed* $S\phi \in D(T)$ *and* $\psi = T \circ S\phi$. *This implies that* $T \circ S$ *is closed. Thus* $T \circ S : X \to Y$ *is closed. Therefore the Closed Graph Theorem implies* $T \circ S \in \mathcal{B}(X, Y)$.

Exercise 1.13. *Let* $T : D(T) \subset X \to Y$ *be a linear operator. If* T *is closed and injective, show that* T^{-1} *is closed.*

1.2. **Closure of an operator. Closable operators.**

Definition 1.14. *Let $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ be two linear operators. We say that B extends A if and only if*

$$D(A) \subseteq D(B)$$
$$B\phi = A\phi, \quad \forall \phi \in D(A).$$

We use the following notation $A \subseteq B$ or $B\big|_{D(A)} = A$.

Example 1.15. *Define the operator*

$$\dot{H}_0 : \mathcal{S}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^n)$$
$$f \mapsto -\Delta f.$$

It is clear that $\dot{H}_0 \subseteq H_0$.

Definition 1.16. *The linear operator $T : D(T) \subset X \to Y$ is* **closable** *<u>if and only if</u> there exists a closed linear operator S with $T \subseteq S$. That is, there exists a closed extension of T.*

Lemma 1.17. *Let $\mathfrak{M}$ be a subspace of $X \times Y$, then $\mathfrak{M}$ is the graph of a linear operator if and only if $\mathfrak{M}$ does not contain points of the form $(0, v)$, $v \neq 0$.*

Proof. Exercise. □

Proposition 1.18. *Let $T : D(T) \subset X \to Y$ be a linear operator. The following affirmations are equivalent:*

(i) *T is closable.*
(ii) *$\overline{G(T)}$ is the graph of a linear operator (closed).*
(iii) *If $\{\phi_n\} \subseteq D(T)$ such that $\phi_n \overset{X}{\to} 0$ and $T\phi_n \overset{Y}{\to} v$, then $v \equiv 0$.*

Proof.
(i) $\implies$ (ii) Let $T : D(T) \subset X \to Y$ be a closable operator, then there exists $S : D(S) \subset X \to Y$ closed such that $T \subseteq S$, that is, $G(T) \subset G(S)$. This implies that

$$\overline{G(T)} \subset \overline{G(S)} = G(S)$$

does not contain points $(0, v)$, $v \neq 0$ by Lemma 1.17. Therefore $\overline{G(T)}$ is the graph of a linear operator which is closed since $\overline{G(T)}$ is closed.

(ii) $\implies$ (iii) If $\{\phi_n\} \subseteq D(T)$ is such that $\phi_n \overset{X}{\to} 0$ and $T\phi_n \overset{Y}{\to} v$, then

$$\underbrace{(\phi_n, T\phi_n)}_{\in G(T)} \overset{X\times Y}{\to} \underbrace{(0, v)}_{\in \overline{G(T)}}$$

This implies that $v \equiv 0$ since $\overline{G(T)}$ is the graph of a linear operator.

(iii) $\implies$ (ii) If $\mathfrak{M} = (0, v) \in \overline{G(T)}$, then $v = 0$ which implies that $\overline{G(T)}$ is the graph of a linear operator.

(ii) $\implies$ (i) Let $S : D(S) \subseteq X \to Y$ be a closed linear operator such that $G(S) = \overline{G(T)}$. Hence

$$G(T) \subseteq \overline{G(T)} = G(S)$$

implies that $T \subseteq S$ is closed and thus T is closable. □

Definition 1.19. *If T is a closable operator, the operator $\overline{T}$ defined by $G(\overline{T}) = \overline{G(T)}$ is called the* **closure** *of T.*

Exercise 1.20. *If $T : D(T) \subset X \to Y$ is closable, show that*

$$D(\overline{T}) = \{\phi \in X : \phi_j \in D(T) \overset{X}{\to} \phi \text{ and } \{T\phi_j\} \text{ is a Cauchy sequence in } Y\}.$$

Example 1.21 (A no closable operator)**.** *Let $X = Y = L^2([0, 1])$, and $\phi \in X$ different from 0. Let*

$$\begin{aligned} T : D(T) = C^0([0, 1]) \subseteq L^2([0, 1]) &\to L^2([0, 1]) \\ f &\mapsto f(1)\phi. \end{aligned}$$

Then T is not closable.

Indeed, suppose that T is closable. Let $f_j(x) = x^j$, then $Tf_j = \phi$ for all $j \in \mathbb{N}$.

On the other hand,

$$\|f_j\|_{L^2} = \Big(\int_0^1 x^{2j}\, dx\Big)^{1/2} = \Big(\frac{1}{2j + 1}\Big)^{1/2} \underset{j\to\infty}{\to} 0$$

Since T is closable then $\phi \equiv 0$ which is a contradiction.

We will see that all differential operator is closable.

Definition 1.22. *Let T be a closed operator, a subspace $\mathfrak{N} \subset D(T)$ is a* **core** *if and only if $\overline{T\big|_{\mathfrak{N}}} = T$, that is, if it is possible to recover T from $\mathfrak{N}$.*

Exercise 1.23. *Show that $C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are core of H_0.*

1.3. **Resolvent, spectrum of an operator.**

Definition 1.24. *Let $T : D(T) \subseteq X \to X$ a linear operator. The* **resolvent** *set of T denoted by $\rho(T)$ is defined by*

$$\rho(T) = \{z \in \mathbb{C} : (T - z)^{-1} \text{ exists and } (T - z)^{-1} \in \mathcal{B}(X)\}.$$

Remark 1.25. *If T is a closed operator we have that*

$$z \in \rho(T) \iff \begin{cases} T - z : D(T) \subseteq X \to X \text{ is injective} \\ T - z : D(T) \subseteq X \to X \text{ is surjective} \end{cases}$$

$$\iff \text{For all } \psi \in X, \text{ there exists a unique } \phi \in D(T) \text{ such that } (T - z)\phi = \psi.$$

Indeed,

$\implies$ *is easy.*

$\impliedby$ *If $T - z$ is $1 - 1$ and surjective, then $(T - z)^{-1} : X \to X$ is closed (exercise). Then applying the closed graph Theorem $(T - z)^{-1} \in \mathcal{B}(X)$.*

Definition 1.26. *The* **spectrum** *of a linear operator T is the set*

$$\sigma(T) = \mathbb{C} \backslash \rho(T).$$

The set of the eigenvalues of T is given by

$$\mathrm{ev}(T) = \{z \in \mathbb{C} : T - z \ \text{ is not } 1 - 1\},$$

i.e.

$$\mathrm{ev}(T) = \{z \in \mathbb{C} : N(T - z) \neq \{0\}\}.$$

Remark 1.27. *We observe that $ev(T) \subseteq \sigma(T)$, but in general the inclusion is strict.*

Example 1.28. *Consider the following operator*

$$T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$$
$$\{x_j\} = (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, \dots).$$

Notice that T is $1 - 1$ but T is not surjective. This in particular implies that

$$ev(T) \subsetneq \sigma(T)$$

since $0 \notin ev(T)$ and $0 \in \sigma(T)$.

Remark 1.29. *There are two possible reasons for $z \in \sigma(T)$.*

(i) *$T - z$ is not $1 - 1$.*
(ii) *$(T - z)^{-1}$ is not defined in the whole X.*

Definition 1.30. *If $z \in \rho(T)$ we define the* **resolvent operator** *by*

$$R_T(z) = (T - z)^{-1}.$$

Remark 1.31. *We observe that*

$$(T - z)R_T(z)\phi = \phi, \quad \forall \phi \in X$$
$$R_T(z)(T - z)\psi = \psi, \quad \forall \psi \in D(T).$$

Exercise 1.32 (An operator without eigenvalues)**.**
Let $D(M) = L^2([-\pi, \pi]) = L^2_{\text{per}}$.

$$M : D(M) \to L^2_{\text{per}}$$
$$f \mapsto Mf(x) = x\, f(x) \quad a.e.\ x \in [-\pi, \pi].$$

Prove that

(i) $M \in \mathcal{B}(L^2([-\pi, \pi]))$;
(ii) $M\phi = \lambda\,\phi \implies \phi = 0$;
(iii) $\sigma(M) = [-\pi, \pi]$.

Exercise 1.33 (Spectrum of H_0 and M_0)**.** *We recall that $M_0 = \mathcal{F}^{-1} \circ H_0 \circ \mathcal{F}$. Show that*

(i) *H_0 and M_0 do not have eigenvalues;*
(ii) $\sigma(H_0) = \sigma(M_0) = \mathbb{R}^+ = [0, \infty)$.

Remark 1.34. *Two linear operators unitarily equivalent have the same spectrum.*

Exercise 1.35. *Consider the operators A_j, $j = 0, 1, 2$, defined by*

$$D(A_0) = H^1([-\pi, \pi]),$$
$$D(A_1) = \{\phi \in \mathcal{D}(A_0)\ /\phi(-\pi) = \phi(\pi)\},$$
$$D(A_2) = \{\phi \in \mathcal{D}(A_1)\ /\ \phi(-\pi) = \phi(\pi) = 0\},$$

and

$$A_j = \frac{1}{i}\frac{d}{dx}, \ j = 0, 1, 2.$$

(i) *Prove that A_j is closed for $j = 0, 1, 2$.*
(ii) *Show that $\sigma(A_0) = \sigma(A_2) = \mathbb{C}$ and $\sigma(A_1) = \mathbb{Z}$.*

Exercise 1.36 (Operator with empty spectrum)**.** *We Define $A^{\pm}$ by*

$$D(A^{\pm}) = \{\phi \in D(A_0)\ :\ \phi(\pm\pi) = 0\},$$
$$A^{\pm}\phi = A_0\phi = \frac{1}{i}\,\phi'.$$

Show that $\sigma(A^{\pm}) = \emptyset$.

Next we recall the following property of the spectrum for bounded operator.

Proposition 1.37. *If $A \in \mathcal{B}(X)$, then the spectrum $\sigma(A) \neq 0$ and $\sigma(A)$ is a compact in $\mathbb{C}$.*

In the case of unbounded operators we only know that $\sigma(T)$ is closed! As a consequence we need the next properties:

Theorem 1.38 (First equation of the resolvent)**.** *Let $T : D(T) \subseteq X \to X$ be a closed linear operator. Suppose that $z, z' \in \rho(T)$, then*

$$R_T(z) - R_T(z') = (z - z')R_T(z) \circ R_T(z').$$

Proof. We have that

$$(T - z') - (T - z) = z - z'.$$

So applying $R_T(z)$ in the above identity, we obtain

$$R_T(z) \circ (T - z') - Id_{D(T)} = (z - z')R_T(z).$$

Now applying $R_T(z')$ on the right, we get the desired equality

$$R_T(z) - R_T(z') = (z - z')R_T(z) \circ R_T(z').$$

□

Corollary 1.39. *It holds that*

$$R_T(z) \circ R_T(z') = R_T(z') \circ R_T(z).$$

Proof. In fact, using

$$R_T(z) - R_T(z') = (z - z')R_T(z) \circ R_T(z')$$

and

$$R_T(z') - R_T(z) = (z' - z)R_T(z') \circ R_T(z)$$

the result follows. □

Theorem 1.40 (Neumann series)**.** *Let X be a Banach space and $A \in \mathcal{B}(X)$ such that $\|A\| < 1$, then $Id - A$ is invertible and*

$$(Id - A)^{-1} = \sum_{j=0}^{\infty} A^j. \tag{1.1}$$

In addition, it holds that

$$\|(Id - A)^{-1}\| \leq \frac{1}{1 - \|A\|}. \tag{1.2}$$

Proof. Let $B = \sum\limits_{j=0}^{\infty} A^j$.

Since

$$\sum_{j=0}^{\infty} \|A^j\| \leq \sum_{j=0}^{\infty} \|A\|^j < \infty,$$

we deduce that the series B is convergent in norm in $\mathcal{B}(X)$ which implies that $B = \sum\limits_{j=0}^{\infty} A^j \in \mathcal{B}(X)$ and for all $n \in \mathbb{N}$ we have that

$$(Id - A) \sum_{j=0}^{n} A^j = \sum_{j=0}^{n} A^j - \sum_{j=1}^{n+1} A^j. = Id - A^{n+1}$$

Making $n \to \infty$ we deduce that $(Id - A)B = Id$.

Similarly we prove that $B(Id - A) = Id$.

Thus $B = (Id - A)^{-1}$ and

$$\|(Id - A)^{-1}\| = \|\sum_{j=0}^{\infty} A^j\| \leq \sum_{j=0}^{n} \|A\|^j = \frac{1}{1 - \|A\|}.$$

□

Corollary 1.41. *If* $T \in \mathcal{G}(X) = \{A \in \mathcal{B}(X); A$ *is invertible,* $A^{-1} \in \mathcal{B}(X)\}$. *Then*

$$B(T, \frac{1}{\|T^{-1}\|}) \subset \mathcal{G}(X).$$

In particular, this implies that $\mathcal{G}(X)$ *is open. In other words, for all* $S \in \mathcal{B}(X)$ *such that* $\|S\| \leq \frac{1}{\|T^{-1}\|}$ *we have that* $T + S \in \mathcal{G}(X)$.

Moreover,

$$\|(T + S)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|S\|\|T^{-1}\|}.$$

Proof. We first notice that

$$T + S = T \circ (Id + T^{-1} \circ S)$$

and

$$\|T^{-1} \circ S\| \leq \|S\|\|T^{-1}\| < 1.$$

This implies that $(Id + T^{-1}S) \in \mathcal{G}(X)$. Hence

$T + S = T \circ (Id + T^{-1} \circ S) \in \mathcal{G}(X)$. (In particular, $\mathcal{G}(X)$ is a group).

In addition,

$$(T + S)^{-1} = (Id + T^{-1}S)^{-1} \circ T^{-1}$$

which implies that

$$\begin{aligned}\|(T+S)^{-1}\| &\leq \|(Id+T^{-1}S)^{-1}\|\|T^{-1}\| \\ &\leq \frac{\|T^{-1}\|}{1-\|T^{-1}\circ S\|} \leq \frac{\|T^{-1}\|}{1-\|S\|\|T^{-1}\|}.\end{aligned}$$

□

Theorem 1.42. *Let $T : D(T) \subseteq X \to X$ be a linear operator not necessarily closed. Then $\rho(T)$ is open in $\mathbb{C}$ and for all $\zeta \in \rho(T)$ and $z \in B(\zeta, \|R_T(\zeta)\|^{-1})$ we have $z \in \rho(T)$ and*

$$R_T(z) = R_T(\zeta)\sum_{j=0}^{\infty} R_T(\zeta)^j (z-\zeta)^j, \quad \forall z \in \mathbb{C} \text{ such that } |z-\zeta| < \|R_T(\zeta)\|^{-1}.$$

Proof. (Idea of the proof) If we know that z and ζ are in $\rho(T)$ then the first equation of the resolvent would imply that

$$R_T(z) - R_T(\zeta) = (z-\zeta)R_T(z)R_T(\zeta)$$

or

$$R_T(z)(Id - (z-\zeta)R_T(\zeta)) = R_T(\zeta).$$

We can see that

$$(Id - (z-\zeta)R_T(\zeta)) \in \mathcal{G}(X) \text{ whenever } \quad |z-\zeta| < \|R_T(\zeta)^{-1}\|.$$

Thus, in this case, it holds that

$$\begin{aligned}R_T(z) &= R_T(\zeta) \circ (Id - (z-\zeta)R_T(\zeta))^{-1} \\ &= R_T(\zeta) \circ \sum_{j=0}^{\infty} (z-\zeta)^j R_T(\zeta)^j.\end{aligned}$$

To prove the theorem we let $\zeta \in \rho(T)$ and $z \in B(\zeta, \|R_T(\zeta)\|^{-1})$. Define

$$F(z) = R_T(\zeta) \circ \sum_{j=0}^{\infty} (z-\zeta)^j R_T(\zeta)^j.$$

We notice that $F(z) \in \mathcal{B}(X)$ since the series $\sum_{j=0}^{\infty} (z-\zeta)^j R_T(\zeta)^j$ converges in norm.

We will show then that $F(z) = R_T(z)$.

For all $\phi \in X$, we have

$$\begin{aligned}(T-z)F(z)\phi &= (T-\zeta)\circ F(z)\phi + (\zeta - z)F(z)\phi \\ &= \Big\{\sum_{j=0}^{\infty}(z-\zeta)^j R_T(\zeta)^j - \sum_{j=0}^{\infty}(z-\zeta)^{j+1}R_T(\zeta)^{j+1}\Big\}\phi \\ &= \phi\end{aligned}$$

Thus

$$(T-z)\circ F(z) = Id. \tag{1.3}$$

Similarly, we get that $F(z)\circ(T-z) = Id$. This implies that $z \in \rho(T)$ and thus

$$R_T(z) = F(z) = R_T(\zeta)\circ\sum_{j=0}^{\infty}(z-\zeta)^j R_T(\zeta)^j.$$

□

Remark 1.43. *Theorem 1.42 tell us that if $T : D(T) \subset X \to X$ is closed, then the map*

$$\begin{aligned} R_T : \rho(T) \subset \mathbb{C} &\to \mathcal{B}(X) \\ z &\mapsto R_T(z)\end{aligned}$$

is a holomorphic function.

Notice that there are several notions to define a holomorphic function $G : \Theta(\text{open}) \subset \mathbb{C} \to \mathcal{B}(\text{X})$.

(i) *$G(z)$ has a power series expansion in terms of each $z_0 \in \Theta$;*
(ii) *$z \mapsto G(z)\phi$ is holomorphic for all $\phi \in X$;*
(iii) *$z \in \Theta \mapsto \langle\psi, G(z)\phi\rangle$ is holomorphic for all $\psi \in X^*$ and for all $\phi \in X$ (G is weakly holomorphic).*

In a Hilbert space, these three notions are equivalent.