Chapter 3
An Introduction to Sobolev Spaces and Pseudo-Differential Operators

In this chapter, we give a brief introduction to the classical Sobolev spaces $H^s(\mathbb{R}^n)$. Sobolev spaces measure the differentiability (or regularity) of functions in $L^2(\mathbb{R}^n)$ and they are a fundamental tool in the study of partial differential equations. We also list some basic facts of the theory of pseudo-differential operators without proof. This is useful to study smoothness properties of solutions of dispersive equations.

3.1 Basics

We begin by defining Sobolev spaces.

Definition 3.1. Let $s \in \mathbb{R}$. We define the *Sobolev space* of order s, denoted by $H^s(\mathbb{R}^n)$, as:

$$H^s(\mathbb{R}^n)=\left\{f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s f(x)=((1+|\xi|^2)^{s/2}\widehat{f}(\xi))^\vee(x) \in L^2(\mathbb{R}^n)\right\}, \tag{3.1}$$

with norm $\|\cdot\|_{s,2}$ defined as:

$$\|f\|_{s,2} = \|\Lambda^s f\|_2. \tag{3.2}$$

Example 3.1 Let $n = 1$ and $f(x) = \chi_{[-1,1]}(x)$. From Example 1.1, we have that $\widehat{f}(\xi) = \sin(2\pi\xi)/(\pi\xi)$. Thus, $f \in H^s(\mathbb{R})$ if $s < 1/2$.

Example 3.2 Let $n = 1$ and $g(x) = \chi_{[-1,1]} * \chi_{[-1,1]}(x)$. In Example 1.2, we saw that

$$\widehat{g}(\xi) = \frac{\sin^2(2\pi\,\xi)}{(\pi\,\xi)^2}.$$

Thus, $g \in H^s(\mathbb{R})$ whenever $s < 3/2$.

Example 3.3 Let $n \geq 1$ and $h(x) = e^{-2\pi|x|}$. From Example 1.4, it follows that

$$\widehat{h}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}\,\frac{1}{(1+|\xi|^2)^{(n+1)/2}}. \tag{3.3}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*, Universitext, DOI 10.1007/978-1-4939-2181-2_3

Using polar coordinates, it is easy to see that $h \in H^s(\mathbb{R}^n)$ if $s < n/2 + 1$. Notice that in this case s depends on the dimension.

Example 3.4 Let $n \geq 1$ and $f(x) = \delta_0(x)$. From Example 1.9, we have $\widehat{\delta_0}(\xi) = 1$. Thus, $\delta_0 \in H^s(\mathbb{R}^n)$ if $s < -n/2$.

From the definition of Sobolev spaces, we deduce the following properties.

Proposition 3.1.

1. *If $s < s'$, then $H^{s'}(\mathbb{R}^n) \subsetneq H^s(\mathbb{R}^n)$.*
2. *$H^s(\mathbb{R}^n)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_s$ defined as follows:*

$$\text{If } f, g \in H^s(\mathbb{R}^n), \text{ then } \langle f, g \rangle_s = \int_{\mathbb{R}^n} \Lambda^s f(\xi) \overline{\Lambda^s g(\xi)} \, d\xi.$$

We can see, via the Fourier transform, that $H^s(\mathbb{R}^n)$ is equal to:

$$L^2(\mathbb{R}^n; (1 + |\xi|^2)^s \, d\xi).$$

3. *For any $s \in \mathbb{R}$, the Schwartz space $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*
4. *If $s_1 \leq s \leq s_2$, with $s = \theta s_1 + (1 - \theta) s_2$, $0 \leq \theta \leq 1$, then*

$$\|f\|_{s,2} \leq \|f\|_{s_1,2}^{\theta} \|f\|_{s_2,2}^{1-\theta}.$$

Proof. It is left as an exercise. □

To understand the relationship between the spaces $H^s(\mathbb{R}^n)$ and the differentiability of functions in $L^2(\mathbb{R}^n)$, we recall Definition 1.2 in the case $p = 2$.

Definition 3.2. A function f is *differentiable* in $L^2(\mathbb{R}^n)$ with respect to the kth variable, if there exists $g \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{f(x + h\, e_k) - f(x)}{h} - g(x) \right|^2 dx \to 0 \text{ when } h \to 0,$$

where e_k has kth coordinate equal to 1 and 0 in the others.

Equivalently (see Exercise 1.9) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, or

$$\int_{\mathbb{R}^n} f(x) \partial_{x_k} \phi(x) \, dx = - \int_{\mathbb{R}^n} g(x) \phi(x) \, dx$$

for every $\phi \in C_0^\infty(\mathbb{R}^n)$ ($C_0^\infty(\mathbb{R}^n)$ being the space of functions infinitely differentiable with compact support).

Example 3.5 Let $n = 1$ and $f(x) = \chi_{(-1,1)}(x)$, then $f' = \delta_{-1} - \delta_1$, where δ_x represents the measure of mass 1 concentrated in x, therefore $f' \notin L^2(\mathbb{R})$.

Example 3.6 Let $n = 1$ and g be as in Example 3.2. Then,

$$\frac{dg}{dx}(x) = \chi_{(-2,0)} - \chi_{(0,2)}, \quad \text{and so} \quad \frac{dg}{dx} \in L^2(\mathbb{R}).$$

With this definition, for $k \in \mathbb{Z}^+$ we can give a description of the space $H^k(\mathbb{R}^n)$ without using the Fourier transform.

Theorem 3.1. *If k is a positive integer, then $H^k(\mathbb{R}^n)$ coincides with the space of functions $f \in L^2(\mathbb{R}^n)$ whose derivatives (in the distribution sense, see (1.42)) $\partial_x^\alpha f$ belong to $L^2(\mathbb{R}^n)$ for every $\alpha \in (\mathbb{Z}^+)^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \le k$.*

In this case, the norms $\|f\|_{k,2}$ and $\sum_{|\alpha|\le k} \|\partial_x^\alpha f\|_2$ are equivalent.

Proof. The proof follows by combining the formula $\widehat{\partial_x^\alpha f}(\xi) = (2\pi i \xi)^\alpha \widehat{f}(\xi)$ (see (1.10)) and the inequalities:

$$|\xi^\beta| \le (1 + |\xi|^2)^{k/2} \le \sum_{|\alpha|\le k} |\xi^\alpha|, \qquad \beta \in (\mathbb{Z}^+)^n, \ |\beta| \le k. \qquad \square$$

Theorem 3.1 allows us to define in a natural manner $H^k(\Omega)$, the Sobolev space of order $k \in \mathbb{Z}^+$ in any subset Ω (open) of $\mathbb{R}^n$. Given $f \in L^2(\Omega)$, we say that $\partial_x^\alpha f$, $\alpha \in (\mathbb{Z}^+)^n$ is the αth partial derivative (in the distribution sense) of f, if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega f \partial_x^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega \partial_x^\alpha f \, \phi \, dx.$$

Then,

$$H^k(\Omega) = \{f \in L^2(\Omega) \ : \ \partial_x^\alpha f (\text{in the distribution sense}) \in L^2(\Omega), \ |\alpha| \le k\}$$

with the norm

$$\|f\|_{H^k(\Omega)} \equiv \Big(\sum_{|\alpha|\le k} \int_\Omega |\partial_x^\alpha f(x)|^2 \, dx\Big)^{1/2}.$$

Example 3.7 For $n = 1$, $b > 0$, and $f(x) = |x|$, one has that $f \in H^1((-b,b))$ and $f \notin H^2((-b,b))$.

The next result allows us to relate "weak derivatives" with derivatives in the classical sense.

Theorem 3.2 (Embedding). *If $s > n/2 + k$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C_\infty^k(\mathbb{R}^n)$, the space of functions with k continuous derivatives vanishing at infinity. In other words, if $f \in H^s(\mathbb{R}^n)$, $s > n/2 + k$, then (after a possible modification of f in a set of measure zero) $f \in C_\infty^k(\mathbb{R}^n)$ and*

$$\|f\|_{C^k} \le c_s \, \|f\|_{s,2}. \tag{3.4}$$

Proof. Case $k = 0$: We first show that if $f \in H^s(\mathbb{R}^n)$, then $\widehat{f} \in L^1(\mathbb{R}^n)$ with

$$\|\widehat{f}\|_1 \leq c_s \|f\|_{s,2}, \quad \text{if } s > n/2. \tag{3.5}$$

Using the Cauchy–Schwarz inequality, we deduce:

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)| d\xi = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|(1 + |\xi|^2)^{s/2} \frac{d\xi}{(1 + |\xi|^2)^{s/2}}$$

$$\leq \|\Lambda^s f\|_2 \left(\int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|^2)^s} \right)^{1/2} \leq c_s \|f\|_{s,2}$$

if $s > n/2$. Combining (3.5), Proposition 1.2, and Theorem 1.1, we conclude that

$$\|f\|_\infty = \|(\widehat{f})^\vee\|_\infty \leq \|\widehat{f}\|_1 \leq c_s \|f\|_{s,2}.$$

Case $k \geq 1$: Using the same argument, we have that if $f \in H^s(\mathbb{R}^n)$ with $s > n/2 + k$, then for $\alpha \in (\mathbb{Z}^+)^n$, $|\alpha| \leq k$, it follows that $\widehat{\partial_x^\alpha f} \in L^1(\mathbb{R}^n)$ and

$$\|\partial_x^\alpha f\|_\infty \leq \|\widehat{\partial_x^\alpha f}\|_1 = \|(2\pi i \xi)^\alpha \widehat{f}\|_1 \leq c_s \|f\|_{s,2}.$$

□

Corollary 3.1. *If $s = n/2 + k + \theta$, with $\theta \in (0, 1)$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $C^{k+\theta}(\mathbb{R}^n)$, the space of C^k functions with partial derivatives of order k Hölder continuous with index θ.*

Proof. We only prove the case $k = 0$, since the proof of the general case follows the same argument. From the formula of inversion of the Fourier transform and the Cauchy–Schwarz inequality we have:

$$|f(x + y) - f(x)| = \left| \int_{\mathbb{R}^n} e^{2\pi i (x \cdot \xi)} \widehat{f}(\xi)(e^{2\pi i (y \cdot \xi)} - 1)\, d\xi \right|$$

$$\leq \Big(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{n/2+\theta} |\widehat{f}(\xi)|^2\, d\xi \Big)^{1/2} \Big(\int_{\mathbb{R}^n} \frac{|e^{2\pi i (y \cdot \xi)} - 1|^2}{(1 + |\xi|^2)^{n/2+\theta}}\, d\xi \Big)^{1/2}.$$

But

$$\int_{\mathbb{R}^n} \frac{|e^{2\pi i (y \cdot \xi)} - 1|^2}{(1 + |\xi|^2)^{n/2+\theta}}\, d\xi$$

$$\leq c \int_{|\xi| \leq |y|^{-1}} |y|^2 |\xi|^2 \frac{d\xi}{(1 + |\xi|^2)^{n/2+\theta}} + 4 \int_{|\xi| \geq |y|^{-1}} \frac{d\xi}{(1 + |\xi|^2)^{n/2+\theta}}$$

$$\leq c|y|^2 \int_0^{|y|^{-1}} \frac{r^{n+1}}{(1+r)^{n+2\theta}}\, dr + 4 \int_{|y|^{-1}}^{\infty} \frac{r^{n-1}}{(1+r)^{n+2\theta}}\, dr \;\leq\; c\,|y|^{2\theta}.$$

If $|y| < 1$, we conclude that $|f(x+y) - f(x)| \leq c\,|y|^{\theta}$. This finishes the proof.□

Theorem 3.3. *If $s \in (0, n/2)$, then $H^s(\mathbb{R}^n)$ is continuously embedded in $L^p(\mathbb{R}^n)$ with $p = 2n/(n-2s)$, i.e., $s = n(1/2 - 1/p)$. Moreover, for $f \in H^s(\mathbb{R}^n)$, $s \in (0, n/2)$,*

$$\|f\|_p \leq c_{n,s}\, \|D^s f\|_2 \leq c\|f\|_{s,2}, \tag{3.6}$$

where

$$D^l f = (-\Delta)^{l/2} f = ((2\pi|\xi|)^l\, \widehat{f})^{\vee}.$$

Proof. The last inequality in (3.6) is immediate, so we just need to show the first one. We define

$$D^s f = g \quad \text{or} \quad f = D^{-s} g = c_{n,s} \left(\frac{1}{|\xi|^s}\, \widehat{g}\right)^{\vee} = \frac{c_{n,s}}{|x|^{n-s}} * g, \tag{3.7}$$

where we have used the result of Exercise 1.14. Thus, by the Hardy–Littlewood–Sobolev estimate (2.10) it follows that

$$\|f\|_p = \|D^{-s} g\|_p = \|\frac{c_{n,s}}{|x|^{n-s}} * g\|_p \leq c_{n,s}\, \|g\|_2 = c\|D^s f\|_2. \tag{3.8}$$

□

We notice from Theorems 3.2 and 3.3, and Corollary 3.1 that the local regularity in H^s, $s > 0$, increases with the parameter s.

Examples 3.1 and 3.3 show that the functions in $H^s(\mathbb{R}^n)$ with $s < n/2$ or $s < n/2 + 1$, respectively, are not necessarily continuous nor C^1. Moreover, let $f \in L^2(\mathbb{R}^n)$ with

$$\widehat{f}(\xi) = \frac{1}{(1+|\xi|)^n\, \log(2+|\xi|)}$$

(which is radial, decreasing, and positive). A simple computation shows that $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$, but $\widehat{f} \notin L^1(\mathbb{R}^n)$ and so $f \notin L^\infty(\mathbb{R}^n)$, since $f(0) = \int \widehat{f}(\xi)d\xi = \infty$ (see also Exercise 3.11(iii)).

To complete the embedding results of the spaces $H^s(\mathbb{R}^n)$, $s > 0$, it remains to consider the case $s = n/2$ (since for $s = k + n/2$, $k \in \mathbb{Z}^+$, the result follows from this one). So, we define the space of functions of the bounded mean oscillation or BMO, introduced by John and Nirenberg [JN].

Definition 3.3. For $f : \mathbb{R}^n \to \mathbb{C}$ with $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we say that $f \in \text{BMO}(\mathbb{R}^n)$ (f has bounded mean oscillation (BMO)) if

$$\|f\|_{\text{BMO}} = \sup_{\substack{x \in \mathbb{R}^n \\ r>0}} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r(x)}|\, dy < \infty, \tag{3.9}$$

where

$$f_{B_r(x)} = \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy.$$

Notice that $\|\cdot\|_{\mathrm{BMO}}$ is a semi-norm since it vanishes for constant functions.

BMO($\mathbb{R}^n$) is a vector space with $L^\infty(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$ since $\|f\|_{\mathrm{BMO}} \le 2\|f\|_\infty$ and $\log|x| \in \mathrm{BMO}(\mathbb{R}^n)$.

Theorem 3.4. *$H^{n/2}(\mathbb{R}^n)$ is continuously embedded in* BMO($\mathbb{R}^n$). *More precisely, there exists $c = c(n) > 0$ such that*

$$\|f\|_{\mathrm{BMO}} \le c\,\|D^{n/2}f\|_2.$$

Proof. Without loss of generality, we assume f real valued. Consider $x \in \mathbb{R}^n$ and $r > 0$.

Let $\phi_r \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi_r \subseteq \{x \mid |x| \le \frac{2}{r}\}$ with $0 \le \phi_r(x) \le 1$ and $\phi_r(x) \equiv 1$ if $|x| < 1/r$, and define

$$f(x) = f_l + f_h = (\widehat{f}\phi_r)^\vee + (\widehat{f}(1-\phi_r))^\vee.$$

We observe that

$$\|f\|_{\mathrm{BMO}} \le \|f_l\|_{\mathrm{BMO}} + \|f_h\|_{\mathrm{BMO}}$$

and $f_l \in H^s(\mathbb{R}^n)$ for any $s > 0$; therefore,

$$f_{l,B_r(x)} = \frac{1}{|B_r(x)|}\int_{B_r(x)} f_l(y)\,dy = f_l(x_0)$$

for some $x_0 \in B_r(x)$, and so for any $y \in B_r(x)$

$$|f_l(y) - f_{l,B_r(x)}| \le 2r\,\|\nabla f_l\|_\infty.$$

Using this estimate we get:

$$\begin{aligned}
\frac{1}{|B_r(x)|}\int_{B_r(x)} \left|f_l(y) - f_{l,B_r(x)}\right|dy &\le \frac{1}{|B_r(x)|^{1/2}}\left(\int_{B_r(x)} |f_l(y) - f_{l,B_r(x)}|^2\,dy\right)^{1/2}\\
&\le 2r\,\|\nabla f_l\|_\infty \le 2r\,\|\widehat{\nabla f_l}\|_1\\
&\le 2r\int_{|\xi|\le 1/2r} |\xi|^{1-n/2}|\xi|^{n/2}|\widehat{f}(\xi)|\,d\xi\\
&\le 2r\left(\int_{|\xi|\le 1/2r} |\xi|^{2-n}\,d\xi\right)^{1/2}\|D^{n/2}f\|_2 \le c\|D^{n/2}f\|_2.
\end{aligned}$$

Also,

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f_h(y) - f_{h,B_r(x)}|dy \le \frac{2}{|B_r(x)|^{1/2}}\|f_h\|_2$$

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$$\leq \frac{2}{|B_r(x)|^{1/2}} \Big(\int_{|\xi|\geq 1/2r} |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2}$$

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$$\leq \frac{c_n}{r^{n/2}} \Big(\int_{|\xi|\geq 1/2r} r^n |\xi|^n |\widehat{f}(\xi)|^2 \, d\xi \Big)^{1/2} \leq \|D^{n/2} f\|_2,$$

which yields the desired result. □

We have shown that $H^s(\mathbb{R}^n)$ with $s > n/2$ is a Hilbert space whose elements are continuous functions. From the point of view of nonlinear analysis, the next property is essential.

Theorem 3.5. *If $s > n/2$, then $H^s(\mathbb{R}^n)$ is an algebra with respect to the product of functions. That is, if $f, g \in H^s(\mathbb{R}^n)$, then $fg \in H^s(\mathbb{R}^n)$ with*

$$\|fg\|_{s,2} \leq c_s \|f\|_{s,2} \|g\|_{s,2}. \tag{3.10}$$

Proof. From the triangle inequality, we have that for every $\xi, \eta \in \mathbb{R}^n$:

$$(1+|\xi|^2)^{s/2} \leq 2^s [(1+|\xi-\eta|^2)^{s/2} + (1+|\eta|^2)^{s/2}].$$

Using this we deduce that

$$\begin{aligned}
|\Lambda^s(fg)| &= |(1+|\xi|^2)^{s/2} \widehat{(fg)}(\xi)| \\
&= (1+|\xi|^2)^{s/2} \Big| \int_{\mathbb{R}^n} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\, d\eta \Big| \\
&\leq 2^s \int_{\mathbb{R}^n} \Big[(1+|\xi-\eta|^2)^{s/2} |\widehat{f}(\xi-\eta)\widehat{g}(\eta)| \\
&\qquad + (1+|\eta|^2)^{s/2} |\widehat{f}(\xi-\eta)\widehat{g}(\eta)| \Big] d\eta \\
&\leq 2^s (|\widehat{\Lambda^s f}| * |\widehat{g}| + |\widehat{f}| * |\widehat{\Lambda^s g}|).
\end{aligned}$$

Thus, taking the L^2-norm and using (1.39) it follows that

$$\|fg\|_{s,2} = \|\Lambda^s(fg)\|_2 \leq c(\|\Lambda^s f\|_2 \|\widehat{g}\|_1 + \|\widehat{f}\|_1 \|\Lambda^s g\|_2). \tag{3.11}$$

Finally, (3.5) assures one that if $r > n/2$, then

$$\begin{aligned}
\|fg\|_{s,2} &\leq c_s(\|f\|_{s,2}\|\widehat{g}\|_1 + \|\widehat{f}\|_1 \|g\|_{s,2}) \\
&\leq c_s(\|f\|_{s,2}\|g\|_{r,2} + \|f\|_{r,2}\|g\|_{s,2}).
\end{aligned} \tag{3.12}$$

Choosing $r = s$, we obtain (3.10). □

The inequality (3.12) is not sharp as the following scaling argument shows. Let $\lambda > 0$ and

$$f(x) = f_1(\lambda x), \quad g(x) = g_1(\lambda x), \quad f_1, g_1 \in \mathcal{S}(\mathbb{R}^n).$$

Then, as $\lambda \uparrow \infty$ the right-hand side of (3.12) grows as λ^{s+r}, meanwhile the left-hand side grows as λ^s. This will not be the case if we replace $\|\cdot\|_{r,2}$ in (3.12) with the $\|\cdot\|_\infty$-norm to get that

$$\|fg\|_{s,2} \le c_s(\|f\|_{s,2}\,\|g\|_\infty + \|f\|_\infty\,\|g\|_{s,2}) \tag{3.13}$$

which in particular shows that for any $s > 0$, $H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is an algebra under the point-wise product.

For $s \in \mathbb{Z}^+$, the inequality (3.13) follows by combining the Leibniz rule for the product of functions and the Gagliardo–Nirenberg inequality:

$$\|\partial_x^\alpha f\|_p \;\le\; c \sum_{|\beta|=m} \|\partial_x^\beta f\|_q^\theta\;\|f\|_r^{1-\theta} \tag{3.14}$$

with $|\alpha| = j$, $c = c(j,m,p,q,r)$, $1/p - j/n = \theta(1/q - m/n) + (1-\theta)1/r$, $\theta \in [j/m, 1]$. For the proof of this inequality, we refer the reader to the reference [Fm].

For the general case $s > 0$, where the usual point-wise Leibniz rule is not available, the inequality (3.13) still holds (see [KPo]). The inequality (3.13) has several extensions, for instance: Let $s \in (0,1)$, $r \in [1,\infty)$, $1 < p_j, q_j \le \infty$, $1/r = 1/p_j + 1/q_j$, $j = 1,2$. Then,

$$\|\Phi^s(fg)\|_r \le c(\|\Phi^s(f)\|_{p_1}\|g\|_{q_1} + \|f\|_{p_2}\|\Phi^s(g)\|_{q_2}),$$

with $\Phi^s = \Lambda^s$ or D^s, (for the proof of this estimate and further generalizations [KPV4], [MPTT], and [GaO]). The extension to the case $r = p_j = q_j = \infty$, $j = 1,2$ was given in [BoLi].

In many applications, the following commutator estimate is often used:

$$\begin{aligned}\sum_{|\alpha|=s} \|[\partial_x^\alpha; g]\,f\|_2 &\equiv \sum_{|\alpha|=s} \|\partial_x^\alpha(gf) - g\partial_x^\alpha f\|_2\\ &\le c_{n,s}\Big(\|\nabla g\|_\infty \sum_{|\beta|=s-1} \|\partial_x^\beta f\|_2 + \|f\|_\infty \sum_{|\beta|=s} \|\partial_x^\beta g\|_2\Big),\end{aligned} \tag{3.15}$$

(see [Kl2]). Similarly, for $s \ge 1$ one has

$$\|[\Lambda^s; g]\,f\|_2 \le c\,(\|\nabla g\|_\infty\|\Lambda^{s-1}f\|_2 + \|f\|_\infty\,\|\Lambda^s g\|_2), \tag{3.16}$$

(see [KPo]).

There are "equivalent" manners to define fractional derivatives without relying on the Fourier transform. For instance:

Definition 3.4 (Stein [S1]). For $b \in (0,1)$ and an appropriate f define

$$\mathcal{D}^b f(x) = \Big(\int \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}}\,dy\Big)^{1/2}. \tag{3.17}$$

Theorem 3.6 (Stein [S1]). *Let* $b \in (0,1)$ *and* $\frac{2n}{(n+2b)} \leq p < \infty$. *Then* f, $D^b f \in L^p(\mathbb{R}^n)$ *if and only if* f, $\mathcal{D}^b f \in L^p(\mathbb{R}^n)$.

Moreover,

$$\|f\|_p + \|D^b f\|_p \sim \|f\|_p + \|\mathcal{D}^b f\|_p.$$

The case $p = 2$ was previously considered in [AS].

For other "equivalent" definitions of fractional derivatives see [Str1].

Finally, to complete our study of Sobolev spaces we introduce the localized Sobolev spaces.

Definition 3.5. Given $f : \mathbb{R}^n \to \mathbb{R}$, we say that $f \in H^s_{loc}(\mathbb{R}^n)$ if for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have $\varphi f \in H^s(\mathbb{R}^n)$. In other words, for any $\Omega \subseteq \mathbb{R}^n$ open bounded $f|_\Omega$ coincides with an element of $H^s(\mathbb{R}^n)$.

This means that f has the sufficient regularity, but may not have enough decay to be in $H^s(\mathbb{R}^n)$.

Example 3.8 Let $n = 1$, $f(x) = x$, and $g(x) = |x|$, then $f \in H^s_{loc}(\mathbb{R})$ for every $s \geq 0$ and $g \in H^s_{loc}(\mathbb{R})$ for every $s < 3/2$.

3.2 Pseudo-Differential Operators

We recall some results from the theory of pseudo-differential operators that we need to describe the local smoothing effect for linear elliptic systems.

The class $S^m = S^m_{1,0}$ of classical symbols of order $m \in \mathbb{R}$ is defined by

$$S^m = \{p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : |p|^{(j)}_{S^m} < \infty,\ j \in \mathbb{N}\}, \tag{3.18}$$

where

$$|p|^{(j)}_{S^m} = \sup\{\|\langle\xi\rangle^{-m+|\alpha|}\partial_\xi^\alpha \partial_x^\beta p(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} : |\alpha+\beta| \leq j\} \tag{3.19}$$

and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

The pseudo-differential operator Ψ_p associated to the symbol $p \in S^m$ is defined by

$$\Psi_p f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\, p(x,\xi)\hat{f}(\xi)\,d\xi, \qquad f \in \mathcal{S}(\mathbb{R}^n). \tag{3.20}$$

Example 3.9 A partial differential operator

$$P = \sum_{|\alpha|\leq N} a_\alpha(x)\partial_x^\alpha,$$

with $a_\alpha \in \mathcal{S}(\mathbb{R}^n)$ is a pseudo-differential operator $P = \Psi_p$ with symbol

$$p(x,\xi) = \sum_{|\alpha|\leq N} a_\alpha(x)(2\pi i\xi)^\alpha \in S^N.$$

Example 3.10 The fractional differentiation operator defined in (3.1) as $\Lambda^\rho = \Psi_{\langle\xi\rangle^\rho}$ is also a pseudo-differential operator with symbol in S^ρ, $\rho \in \mathbb{R}$.

The collection of symbol classes S^m, $m \in \mathbb{R}$, is in some cases closed under composition, adjointness, division, and square root operations. This is not the case for polynomials in ξ, and sometimes this closure allows one to construct approximate inverses and square roots of pseudo-differential operators.

Next, we list some properties of pseudo-differential operators whose proofs can be found for instance in [Kg].

Theorem 3.7 (Sobolev boundedness). *Let $m \in \mathbb{R}$, $p \in S^m$, and $s \in \mathbb{R}$. Then, Ψ_p extends to a bounded linear operator from $H^{m+s}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. Moreover, there exist $j = j(n;m;s) \in \mathbb{N}$ and $c = c(n;m;s)$ such that*

$$\|\Psi_p f\|_{H^s} \leq c\,|p|^{(j)}_{S^m}\,\|f\|_{H^{m+s}}. \tag{3.21}$$

Theorem 3.8 (Symbolic calculus). *Let $m_1, m_2 \in \mathbb{R}$, $p_1 \in S^{m_1}$, $p_2 \in S^{m_2}$. Then, there exist $p_3 \in S^{m_1+m_2-1}$, $p_4 \in S^{m_1+m_2-2}$, and $p_5 \in S^{m_1-1}$ such that*

$$\begin{aligned} \Psi_{p_1}\Psi_{p_2} &= \Psi_{p_1 p_2} + \Psi_{p_3}, \\ \Psi_{p_1}\Psi_{p_2} - \Psi_{p_2}\Psi_{p_1} &= \Psi_{-i\{p_1,p_2\}} + \Psi_{p_4}, \\ (\Psi_{p_1})^* &= \Psi_{\bar{p}_1} + \Psi_{p_5}, \end{aligned} \tag{3.22}$$

where $\{p_1, p_2\}$ denotes the Poisson bracket, i.e.,

$$\{p_1, p_2\} = \sum_{j=1}^{n} (\partial_{\xi_j} p_1\, \partial_{x_j} p_2 - \partial_{x_j} p_1\, \partial_{\xi_j} p_2), \tag{3.23}$$

and such that for any $j \in \mathbb{N}$ there exist $j' \in \mathbb{N}$ and $c_1 = c_1(n;m_1;m_2;j)$, $c_2 = c_2(n;m_1;j)$ such that

$$\begin{aligned} |p_3|^{(j)}_{S^{m_1+m_2-1}} + |p_4|^{(j)}_{S^{m_1+m_2-2}} &\leq c_1\,|p_1|^{(j')}_{S^{m_1}}\,|p_2|^{(j')}_{S^{m_2}} \\ |p_5|^{(j')}_{S^{m_1-1}} &\leq c_2\,|p_1|^{(j')}_{S^{m_1}}. \end{aligned}$$

Remark 3.1.

(i) (3.22) tell us that the "principal symbol" of the commutator $[\psi_{p_1}; \psi_{p_2}]$ is given by the formula in (3.23).

(ii) It is useful for our purpose to consider the class of symbols $S^{m,N} = S^{m,N}_{1,0}$ defined as $p(x,\xi) \in C^N(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|p|^{(N)}_{S^m} < \infty, \quad \text{with } |p|^{(N)}_{S^m} \text{ defined in (3.19).} \tag{3.24}$$

For N sufficiently large the results in Theorem 3.7 extend to the class $S^{m,N}$.

3.3 The Bicharacteristic Flow

In this section, we introduce the notion of bicharacteristic flow. This plays a key role in the study of linear variable coefficients Schrödinger equations and in the well-posedness of the initial value problem (IVP) associated to the quasilinear case as we can see in the next and the last chapters.

Let $\mathcal{L} = \partial_{x_j} a_{jk}(x)\partial_{x_k}$ be an elliptic self-adjoint operator, that is, $(a_{jk}(x))_{jk}$ is a $n \times n$ matrix of functions $a_{jk} \in C_b^\infty$, real, symmetric, and positive definite, i.e., $\exists\, \nu > 0$ such that $\forall\, x, \xi \in \mathbb{R}^n$,

$$\nu^{-1}\|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leq \nu\|\xi\|^2. \tag{3.25}$$

Let h_2 be the principal symbol of $\mathcal{L}$, i.e.,

$$h_2(x,\xi) = -\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k. \tag{3.26}$$

The bicharacteristic flow is the flow of the Hamiltonian vector field:

$$H_{h_2} = \sum_{j=1}^{n} [\partial_{\xi_j} h_2 \cdot \partial_{x_j} - \partial_{x_j} h_2 \cdot \partial_{\xi_j}] \tag{3.27}$$

and is denoted by $(X(s;x_0,\xi_0), \Xi(s;x_0,\xi_0))$, i.e.,

$$\begin{cases} \dfrac{d}{ds}X_j(s;x_0,\xi_0) = -2\sum\limits_{k=1}^{n} a_{jk}(X(s;x_0,\xi_0))\,\Xi_k(s;x_0,\xi_0), \\ \dfrac{d}{ds}\Xi_j(s;x_0,\xi_0) = \sum\limits_{k,l=1}^{n} \partial_{x_j} a_{lk}(X(s;x_0,\xi_0))\,\Xi_k(s;x_0,\xi_0)\,\Xi_l(s;x_0,\xi_0) \end{cases} \tag{3.28}$$

for $j = 1, \ldots, n$, with

$$(X(0;x_0,\xi_0), \Xi(0;x_0,\xi_0)) = (x_0,\xi_0). \tag{3.29}$$

The bicharacteristic flow exists in the time interval $s \in (-\delta, \delta)$ with $\delta = \delta(x_0,\xi_0)$, and $\delta(\cdot)$ depending continuously on (x_0,ξ_0).

The bicharacteristic flow preserves h_2, i.e.,

$$\frac{d}{ds}h_2(X(s;x_0,\xi_0), \Xi(s;x_0,\xi_0)) = 0,$$

so the ellipticity hypothesis (3.25) gives

$$\nu^{-2}\|\xi_0\|^2 \leq \|\Xi(s;x_0,\xi_0)\|^2 \leq \nu^2\|\xi_0\|^2, \tag{3.30}$$

and hence $\delta = \infty$.

In the case of constant coefficients, $h_2(x,\xi) = -|\xi|^2$, the bicharacteristic flow is given by $(X, \Xi)(\xi, x_0, \xi_0) = (x_0 - 2s\xi_0, \xi_0)$.

For general symbol $h(x,\xi)$, the bicharacteristic flow is defined as:

$$\begin{cases} \dfrac{dX}{ds} = \partial_\xi h(X, \Xi) \\ \dfrac{d\Xi}{ds} = -\partial_x h(X, \Xi). \end{cases} \tag{3.31}$$

In applications, the notion of the bicharacteristic flow

$$t \mapsto (X(t; x_0, \xi_0),\ \Sigma(t; x_0, \xi_0)) \tag{3.32}$$

being nontrapping arises naturally.

Definition 3.6. A point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ is nontrapped forward (respectively, backward) by the bicharacteristic flow if

$$\|X(t;\, x_0,\, \xi_0)\| \to \infty \text{ as } t \to \infty \quad (\text{resp, } t \to -\infty). \tag{3.33}$$

If each point $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n - \{0\}$ is nontrapped forward, then the bicharacteristic flow is said to be nontrapping.

In particular, if one assumes that the "metric" $(a_{jk}(x))$ in (3.26) possesses an "asymptotic flat property," for example,

$$|\partial_x^\alpha(a_{jk}(x) - \delta_{jk})| \leq \frac{c_\alpha}{|x|^{1+\epsilon(\alpha)}},\quad \epsilon(\alpha) > 0,\quad 0 \leq |\alpha| \leq m = m(n), \tag{3.34}$$

then it suffices to have that for each $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ and for each $\mu > 0$ there exists $\hat{t} = \hat{t}(\mu;\, x_0, \xi_0) > 0$ such that

$$\|X(\hat{t};\, x_0,\, \xi_0)\| \geq \mu$$

to guarantee that the bicharacteristic flow is nontrapping.

The next result shows that the Hamiltonian vector field is differentiation along the bicharacteristics.

Lemma 3.1. *Let $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then,*

$$(H_{h_2}\phi)(x,\xi) = \partial_s[\phi(X(s;x,\xi), \Xi(s;x,\xi))]|_{s=0} = \{h_2, \phi\}. \tag{3.35}$$

Notice that $-i\{h_2, \phi\}$ is the principal symbol of the commutator $[\psi_{h_2}, \psi_\phi]$ (see 3.22).

Proof. By the chain rule,

$$\begin{aligned} \partial_s[\phi(X(s;x,\xi), \Xi(s;x,\xi))] &= (\nabla_x\phi)(X(s;x,\xi), \Xi(s;x,\xi)) \cdot \partial_s X(s;x,\xi) \\ &\quad + (\nabla_\xi\phi)(X(s;x,\xi), \Xi(s;x,\xi)) \cdot \partial_s \Xi(s;x,\xi) \\ &= (\nabla_x\phi \cdot \nabla_\xi h_2)(X(s;x,\xi), \Xi(s;x,\xi)) \\ &\quad - (\nabla_\xi\phi \cdot \nabla_x h_2)(X(s;x,\xi), \Xi(s;x,\xi)). \end{aligned}$$

Setting $s = 0$, the lemma follows. □

3.4 Exercises

3.1 Prove that for any $k \in \mathbb{Z}^+$ and any $\theta \in (0,1)$

$$\chi_{(-1,1)} \overset{k \text{ times}}{* \cdots *} \chi_{(-1,1)}(x) \in C_0^{k-1,\theta}(\mathbb{R}) \backslash C^k(\mathbb{R}).$$

3.2 Prove Proposition 3.1.

3.3 Let $f_n : \mathbb{R}^n \to \mathbb{R}$ with $f_n(x) = e^{-2\pi |x|}$.

(i) Prove that $f_1 * f_1(x) = \dfrac{e^{-2\pi |x|}}{2\pi}(1 + 2\pi |x|)$.
Hint: Use an explicit computation or Exercise 1.1(ii).

(ii) Show that $f_1 * f_1(x) \in C^2(\mathbb{R})$, but is not in $C^3(\mathbb{R})$.

(iii) Prove that $f_n * f_n \in C_\infty^{n+1}(\mathbb{R}^n)$.

(iv) More general, prove that if $g \in H^{s_1}(\mathbb{R}^n)$ and $h \in H^{s_2}(\mathbb{R}^n)$, then $g * h \in C_\infty^{[s_1+s_2]}(\mathbb{R}^n)$ (where $[\cdot]$ denotes the greatest integer function.)

3.4 Let $\phi(x) = e^{-|x|}$, $x \in \mathbb{R}$:

(i) Prove that

$$\phi(x) - \phi''(x) = 2\delta, \tag{3.36}$$

(a) in the distribution sense, i.e., $\forall \varphi \in C_0^\infty(\mathbb{R})$,

$$\int \phi(x)(\varphi(x) - \varphi''(x))\,dx = 2\varphi(0),$$

(b) by taking the Fourier transform in (3.36).

(ii) Prove that given $g \in L^2(\mathbb{R})$ (or $H^s(\mathbb{R})$) the equation:

$$\left(1 - \frac{d^2}{dx^2}\right) f = g$$

has solution $f = \frac{1}{2} e^{-|\cdot|} * g \in H^2(\mathbb{R})$ (or $H^{s+2}(\mathbb{R})$).

3.5 Show that if $k \in \mathbb{Z}^+$ and $p \in [1, \infty)$, then

$$F_{k,p}(\mathbb{R}^n) = L_k^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

is a Banach algebra with respect to point-wise product of functions. Moreover, if $f, g \in F_{k,p}$, then

$$\|fg\|_{k,p} \leq c_k(\|f\|_{k,p}\|g\|_\infty + \|f\|_\infty \|g\|_{k,p}). \tag{3.37}$$

Notation:

$$L_k^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} : \partial^\alpha f \text{ (distribution sense) } \in L^p,\ |\alpha| \leq k\},$$

whose norm is defined as:

$$\|f\|_{k,p} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_p.$$

Observe that when $p = 2$ one has $L^p_k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$.
More generally, we define

$$L^p_s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^p(\mathbb{R}^n) \text{ for } s \in \mathbb{R}, \text{ with } \|f\|_{s,p} = \|(1-\Delta)^{s/2}f\|_p. \tag{3.38}$$

Hint: From Leibniz formula and Hölder's inequality it follows that (assume $n = 1$ to simplify)

$$\|(fg)^{(k)}\|_p \le \sum_{j=0}^{k} c_j \|f^{(k-j)}\|_{p_{j_1}} \|g^{(j)}\|_{p_{j_2}}, \quad \text{with } \frac{1}{p} = \frac{1}{p_{j_1}} + \frac{1}{p_{j_2}}.$$

Combine the Gagliardo–Nirenberg inequality (3.14):

$$\|h^{(k-j)}\|_{p_j} \le c\|h^{(k)}\|_p^{\theta}\|h\|_\infty^{1-\theta}, \quad \theta = \theta(n,k,j,p_j),$$

with Young's inequality (if $1/p + 1/p' = 1$ with $p > 1$, then $ab \le a^p/p + b^{p'}/p'$) to get the desired result (3.37).

3.6 Extend the result of Theorem 3.3 to the spaces $L^p_s(\mathbb{R}^n)$, i.e., if $f \in L^p_s(\mathbb{R}^n)$, $0 < s < n/p$, then $f \in L^r(\mathbb{R}^n)$ with $s = n(\frac{1}{p} - \frac{1}{q})$, and

$$\|f\|_r \le c_{n,s}\|D^s f\|_p \le c_{n,s}\|f\|_{s,p}. \tag{3.39}$$

3.7 (i) Prove that if $1 < p < \infty$ and $b \in (0,1)$, then

$$\|\Lambda^b f\|_p \sim \|f\|_p + \|D^b f\|_p.$$

Hint: Use Theorem 2.8.

(ii) Given any $s \in \mathbb{R}$ find $f_s \in H^s(\mathbb{R})$ such that $f_s \notin H^{s'}(\mathbb{R})$ for any $s' > s$.
Hint:

(a) Notice that it suffices to find f_0.
(b) Show that if $g \in L^2(\mathbb{R})$ and $g \notin L^p(\mathbb{R})$ for any $p > 2$, then one can take $f_0 = g$.
(c) Use (b) to find f_0.

3.8 Show that if $f \in H^s(\mathbb{R}^n)$, $s > n/2$, with $\|f\|_{n/2,2} \le 1$, then

$$\|f\|_\infty \le c\,[1 + \log(1 + \|f\|_{s,2})]^{1/2}$$

with $c = c(s,n)$, see [BGa].

3.9 Prove the following inequalities:

(i) If $s > n/2$, then

$$\|f\|_\infty \le c_{n,s}\|f\|_2^{1-n/2s}\|D^s f\|_2^{n/2s}.$$

(ii) If $s > n/p$, $1 < p < \infty$, then

$$\|f\|_\infty \leq c_{n,s,p} \|f\|_p^{1-n/ps} \|D^s f\|_p^{n/ps}.$$

(iii) Prove Gagliardo–Nirenberg inequality (3.14) for p even integer, $m = 2$, $j = 2$, and $q, r \in (1, \infty)$ such that $1/q + 1/r = 2/p$.

(iv) Combine Exercises 2.10 and 2.11, and Theorem 2.6 to prove the Gagliardo–Nirenberg inequality in the general case.

3.10 ([AS]). Using Definition 3.4:

(i) Prove that for $b \in (0, 1)$

$$\|D^b f\|_2 = c_n \|\mathcal{D}^b f\|_2. \tag{3.40}$$

(ii) Prove that

$$\mathcal{D}^b(fg)(x) \leq \|f\|_\infty \mathcal{D}^b g(x) + |g(x)| \mathcal{D}^b f(x) \tag{3.41}$$

and

$$\|\mathcal{D}^b(fg)\|_2 \leq \|f\mathcal{D}^b g\|_2 + \|g\mathcal{D}^b f\|_2. \tag{3.42}$$

(iii) Let $F \in C_b^1(\mathbb{R} : \mathbb{R})$, $F(0) = 0$. Show that

$$\|D^b(F(f))\|_2 \leq \|F'\|_\infty \|D^b f\|_2.$$

Hint: Apply part (i).

3.11 (i) Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, be such that $f(x_0^+)$, $f(x_0^-)$ exist and $f(x_0^+) \neq f(x_0^-)$ for some x_0. Prove that $f \notin L^p_{1/p}(\mathbb{R})$.

(ii) Let $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(x) = 1$ if $|x| \leq 1$ and $\varphi(x) = 0$ if $|x| > 2$. Let $a, b \in (0, 1)$. Prove that $|x|^a \varphi(x) \in H^b(\mathbb{R})$ if and only if $b < a + 1/2$.

(iii) Let $\alpha \in (0, 1/2)$. Prove that

$$|\log|x||^\alpha \chi_{\{|x| \leq 1/10\}} + \frac{10}{9}(1 - |x|)\chi_{\{1/10 \leq |x| \leq 1\}} \in H^1(\mathbb{R}^2) - L^\infty(\mathbb{R}^2).$$

3.12 (Sobolev's inequality for radial functions) Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be a radial function, i.e., $f(x) = f(y)$ if $|x| = |y|$. Show that f satisfies

$$|f(x)| \leq c_n |x|^{(2-n)/2} \|\nabla f\|_2.$$

3.13 (Hardy's inequalities (see Exercise 1.5))

(i) Let $1 \leq p < \infty$. If $f \in L_1^p(\mathbb{R}^n)$, then

$$\left\| \frac{|f(\cdot)|}{|x|} \right\|_p \leq \frac{p}{n-p} \|\nabla f\|_p. \tag{3.43}$$

(ii) Let $1 \le p < \infty$, $q < n$, and $q \in [0, p]$. If $f \in L^p_1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^q}\,dx \le \left(\frac{p}{n-q}\right)^q \|f\|_p^{p-q}\|\nabla f\|_p^q. \tag{3.44}$$

Hint: Assume that $f \in C_0^\infty(\mathbb{R}^n)$. For (i), write $\||\cdot|^{-1} f\|_p^p$ in spherical coordinates, use integration by parts in the radial variable and Hölder inequality to get the result. For (ii), assume $p > q$, and apply (3.43) to $|x|^{-1} g(x)$ with $g(x) = |f(x)|^{p/q}$.

3.14 Prove Heisenberg's inequality. If $f \in H^1(\mathbb{R}^n) \cap L^2(|x|^2\,dx)$, then

$$\|f\|_2^2 \le \frac{2}{n}\|x_j f\|_2\|\partial_{x_j} f\|_2 = \frac{4\pi}{n}\|x_j f\|_2\|\xi_j \widehat{f}\|_2 \le \frac{2}{n}\|xf\|_2\|\nabla f\|_2. \tag{3.45}$$

Hint: Use the density of $\mathcal{S}(\mathbb{R}^n)$ and integration by parts to obtain the identity

$$\|f\|_2^2 = -\frac{1}{n}\int x_j\partial_{x_j}(|f(x)|^2)\,dx.$$

3.15 Denote $u = u(x,t)$, the solution of the IVP associated to the inviscid Burgers' equation:

$$\begin{cases} \partial_t u + u\partial_x u = 0, \\ u(x,0) = u_0(x) \in C_0^\infty(\mathbb{R}), \end{cases} \tag{3.46}$$

$t, x \in \mathbb{R}$. Prove that for every $T > 0$,

$$u \in C^\infty(\mathbb{R} \times [-T,T]) \quad \text{or} \quad u \notin C^1(\mathbb{R} \times [-T,T]).$$

Hint: Combine the commutator estimate (3.16) and integration by parts to obtain the energy estimate

$$\frac{d}{dt}\|u(t)\|_{k,2} \le c_k\|\partial_x u(t)\|_\infty\|u(t)\|_{k,2} \quad \text{for all } k \in \mathbb{Z}^+. \tag{3.47}$$

3.16 Let $P(x,\partial_x) = \sum_{|\alpha|\le m_1} a_\alpha(x)\partial_x^\alpha$ and $Q(x,\partial_x) = \sum_{|\alpha|\le m_2} b_\alpha(x)\partial_x^\alpha$ be two differential operators. Check the properties stated in Theorem 3.8 for P and Q.

3.17 (i) If $\Lambda = (1-\Delta)^{1/2}$ and $y \in \mathbb{R}$, show that the symbol $p = p(\xi)$ of Λ^{iy}, $p(\xi) = (1+|\xi|^2)^{iy/2} \in S_0$, and

$$|p|^j_{S^0} \le c_n(1+|y|)^j.$$

(ii) Show that if $p = p(x,\xi) \in S^0 = S^0_{1,0}$, then $e^{p(x,\xi)} \in S^0 = S^0_{1,0}$.

3.18 Prove that the bicharacteristic flow in (3.28) $(X(s;x_0,\xi_0),\ \Xi_k(s;x_0,\xi_0))$ satisfies

(i) $X(s; x_0, \rho\,\xi_0) = X(\rho\, s; x_0, \xi_0)$,
(ii) $\Xi_k(s; x_0, \rho\,\xi_0) = \rho\,\Xi_k(\rho\, s; x_0, \xi_0)$.

Hint: Use the homogeneity of $h_2(x, \xi) = -a_{jk}(x)\,\xi_j \xi_k$.

3.19 Prove that if Ψ_p is a pseudo-differential operator with symbol $p \in S^0$, then for any $b \in \mathbb{R}$,

$$\|\Psi_p f\|_{L^2(\langle x\rangle^b dx)} \leq c_{k,n}\, \|f\|_{L^2(\langle x\rangle^b dx)}, \tag{3.48}$$

where

$$\|g\|_{L^2(\langle x\rangle^b dx)} = \left(\int |g(x)|^2 \langle x\rangle^b\, dx\right)^{1/2}$$

and

$$\langle x\rangle = (1 + |x|^2)^{1/2}. \tag{3.49}$$

Hint:

(i) Follow an argument similar to that given in the proof of Theorem 2.1 to show that it suffices to establish (3.48) for $b = 4k$, $k \in \mathbb{Z}$.
(ii) Consider the case $b = -4k$, $k \in \mathbb{Z}^+$, and show that (3.48) is equivalent to

$$\left\|\frac{1}{\langle x\rangle^{2k}}\, \Psi_p\left(\langle x\rangle^{2k} g\right)\right\|_2 \leq c\, \|g\|_2. \tag{3.50}$$

(iii) Obtain (3.50) by combining integration by parts, Theorems 3.7 and 3.8.
(iv) Finally, prove the case $b = 4k$, $k \in \mathbb{Z}^+$, by duality.

3.20 Let $a, b > 0$. Assume that $\Lambda^a f = (1 - \Delta/4\pi^2)^{a/2} f \in L^2(\mathbb{R}^2)$ (i.e., $f \in H^a(\mathbb{R}^n)$) and $\langle x\rangle^b f \in L^2(\mathbb{R}^n)$ (see 3.49). Prove that for any $\theta \in (0, 1)$,

$$\|\Lambda^{(1-\theta)a}(\langle x\rangle^{\theta b} f)\|_2 \leq c_{a,b,n}\, \|\langle x\rangle^b f\|_2^{\theta} \|\Lambda^a f\|_2^{1-\theta}.$$

Hint: Combine the three lines theorem, Exercises 3.17 part (i) and (3.19).