Teoria Espectral

1. Introduction

Spectral Theory is the branch of analysis concentrated in the study of properties of linear operators in infinite dimension.

We will start by introducing some definitions and notations.

Definition 1.1. *Let X, Y be Banach spaces (real or complex). A* **linear operator** *is an application $A : D(A) \subset X \to Y$ such that*

(i) *$D(A)$ is a vector subspace of X.*
(ii) *$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$, for all $x, y \in D(A)$ and for all $\alpha, \beta \in \mathbb{R}(or\,\mathbb{C})$.*

Notation

- The domain of the operator A is denoted by $D(A)$.
- The image or range of the operator A, is defined by
$$R(A) = \{Ax : x \in D(A)\}.$$
- The kernel of the operator A, is defined by
$$Ker(A) = N(A) = A^{-1}(\{0\}) = \{x \in D(A) : Ax = 0\}.$$
- The graph of of the operator A is defined by
$$G(A) = Graph(A) = \{(x, Ax) : x \in D(A)\}.$$

Remark 1.2. *Notice that $R(A)$ is a subspace of Y, $N(A)$ is a subspace of X and $G(A)$ is a subspace of $X \times Y$.*

Definition 1.3. *Let A be a linear operator. $A : D(A) \subset X \to Y$ is called bounded if and only if there exists $c > 0$ such that*

$$\|Ax\|_Y \leq c\,\|x\|_X, \quad \text{for all } x \in D(A). \tag{1.1}$$

The norm of the operator A is defined as

$$\|A\| = \|A\|_{X,Y} = \inf\{c > 0 : (1.1) \quad holds\}.$$

The set of all bounded operators from X to Y is given by

$$\mathcal{B}(X, Y) = \{A : D(A) \subset X \to Y : A \text{ is bounded}\}.$$

In the case $X = Y$, we use the notation $\mathcal{B}(X) = \mathcal{B}(X, X)$.

Exercise 1.4. *Show that the following statements are equivalent.*

(i) *A is bounded.*
(ii) *A is continuous.*
(iii) *A is continuous at the origin.*

Exercise 1.5. *Show that* $\|\cdot\|$ *defines a norm on* $\mathcal{B}(X,Y)$.

Exercise 1.6. *Prove that*

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

Exercise 1.7. *Show that*

$$\|A\cdot B\| \leq \|A\|\|B\|, \text{ for all } A, B \in \mathcal{B}(X,Y).$$

Example 1.8. *Let* A *be the multiplication operator defined as*

$$A : L^2([0,1]) \to L^2([0,1])$$
$$f \mapsto Af : [0,1] \to \mathbb{C}$$
$$x \to xf(x).$$

$A \in \mathcal{B}(L^2([0,1]))$.
In fact,

$$\|Af\|_{L^2}^2 = \int_0^1 |xf(x)|^2\, dx \leq \sup_{x\in[0,1]} |x|^2 \int_0^1 |f(x)|^2\, dx = \|f\|_{L^2}^2.$$

Example 1.9. *Let* M *be the operator defined as*

$$M : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$f \mapsto Mf : \mathbb{R} \to \mathbb{C}$$
$$x \to xf(x).$$

Let

$$g(x) = \begin{cases} 0, & x \in (-\infty, 1), \\ \\ \dfrac{1}{x}, & x \in (1,\infty), \end{cases}$$

then

$$\int_{\mathbb{R}} |g(x)|^2\, dx = \int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^\infty = 1.$$

Thus $g \in L^2(\mathbb{R})$.
On the other hand,

$$Mg(x) = \begin{cases} 0, & x \in (-\infty, 1), \\ \\ 1, & x \in (1,\infty), \end{cases} \notin L^2(\mathbb{R}).$$

We see that the operator M *is not well defined on the space* $L^2(\mathbb{R})$.

We shall define the domain of M as follows

$$D(M) = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\} \subsetneq L^2(\mathbb{R}).$$

In addition, M is not bounded in $L^2(\mathbb{R})$. Indeed,

Let

$$\psi_n(x) = \begin{cases} 0, & x \leq n, \\ \dfrac{1}{x}, & n < x < n+1, \\ 0, & x \geq n+1. \end{cases}$$

Then

$$\|\psi_n\|_{L^2}^2 = \int_n^{n+1} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_n^{n+1} = \frac{1}{n} - \frac{1}{n+1}$$

Thus

$$\|\psi_n\|_{L^2} = \left(\frac{1}{n(n+1)}\right)^{1/2}$$

and

$$\|M\psi_n\|_{L^2}^2 = \int_n^{n+1} dx = 1.$$

Therefore $\psi_n \in D(M)$.

If $M \in \mathcal{B}(L^2(\mathbb{R}))$, then there would exist $c > 0$ such that

$$\|Mf\|_{L^2} \leq c\,\|f\|_{L^2}, \quad \textit{for all } f \in L^2(\mathbb{R}).$$

But then

$$\|M\psi_n\|_{L^2} = 1 \leq c\,\|\psi_n\|_{L^2} = c\left(\frac{1}{n(n+1)}\right)^{1/2} \to 0 \ \textit{as}\ n \to \infty.$$

Which is a contradiction!

Thus $M \notin \mathcal{B}(L^2(\mathbb{R}))$.

Remark 1.10. *What we observe in the previous examples that difficulties come from problems with the domain.*

Example 1.11. *Consider two linear operators*

$$A : D(A) \subset X \to Y$$
$$B : D(B) \subset X \to Y.$$

How can we define the operator $A + B$?

We can try the natural definition

$$A + B : D(A) \cap D(B) \subset X \to Y$$
$$x \longrightarrow Ax + Bx.$$

However there exist dense subspaces with intersection $\{0\}$ in $L^2(\mathbb{R})$ for instance. (add example)

Definition 1.12. *Let $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ be two linear operators. We said that the linear operator B* **extends** *the linear operator A denoted by $A \subseteq B$, if*

$$D(A) \subset D(B) \quad \text{and} \quad Bx = Ax, \quad \text{for } x \in D(A).$$

We call B an **extension** *of A.*

Theorem 1.13. *Let $A : D(A) \subset X \to Y$ be a bounded linear operator. Then there exists a unique extension $\bar{A}$ of A, $\bar{A} : \overline{D(A)} \subset X \to Y$ such that $\|\bar{A}\| = \|A\|$.*

In particular, if $\overline{D(A)} = X$ then A extends over all X.

Proof. Let $x \in \overline{D(A)}$ there exists a sequence $\{x_n\} \subset D(A)$ suc that $x_n \to x$ and

$$\|Ax_n - Ax_m\| = \|A(x_n - x_m)\| \leq \|A\|\|x_n - x_m\| \to 0 \quad \text{as } n, m \to \infty.$$

Then $\{Ax_n\}$ is a Cauchy sequence in the Banach space Y. Hence there exists $y \in Y$ such that $Ax_n \to y$ as $n \to \infty$.

We define $\bar{A}x = y$.

Observe that y does not depend on the sequence $\{x_n\}$ such that $x_n \to x$. Indeed, if we have another sequence $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$ and $A\tilde{x}_n \to \tilde{y}$, then

$$\|y - \tilde{y}\| \leq \|y - Ax_n\| + \|A\|\|x_n - \tilde{x}_n\| + \|A\tilde{x}_n - \tilde{y}\| \to 0$$

as $n \to \infty$. Thus $y = \tilde{y}$.

Hence $\bar{A}\big|_{D(A)} = A$, $\bar{A}$ is linear on $\overline{D(A)}$ and if $x \in \overline{D(A)}$, with $x_n \to x$, then

$$\|\bar{A}x\| = \lim_{n\to\infty} \|Ax_n\| \leq \|A\| \lim_{n\to\infty} \|x_n\| = \|A\|\|x\|,$$

which implies that $\bar{A} \in \mathcal{B}(X, Y)$ and $\|\bar{A}\| \leq \|A\|$.

On the other hand, since $\bar{A}\big|_{D(A)} = A$ for any $x \in D(A)$, it follows that

$$\|Ax\| = \|\bar{A}x\| \leq \|\bar{A}\|\|x\|.$$

This inequality implies that $\|A\| \leq \|\bar{A}\|$. Therefore $\|A\| = \|\bar{A}\|$. $\square$

Example 1.14. *The Fourier transform in $L^1(\mathbb{R}^n)$.*

Definition 1.15. *Let $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is defined by*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx, \quad \text{for any } \xi \in \mathbb{R}^n,$$

where $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$.

We use the notation $\mathcal{F}f = \widehat{f} : \xi \in \mathbb{R}^n \to \widehat{f}(\xi)$ *to the denote the Fourier transform of* f*. This map is clearly linear.*

Proposition 1.16. *If* $f \in L^1(\mathbb{R}^n)$*, then* $\widehat{f} \in C^0_\infty(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ *and* $\mathcal{F} \in \mathcal{B}(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)$*.*

Here $C^0_\infty(\mathbb{R}^n)$ denotes the space of continuous functions which vanish as $|x| \to \infty$.

Proof. For any $\xi \in \mathbb{R}^n$

$$|\widehat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\, dx = \|f\|_{L^1}.$$

which implies

$$\|\widehat{f}\|_{L^\infty} \le \|f\|_{L^1}.$$

Thus $\mathcal{F} \in \mathcal{B}(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$.

Afirmmation $\widehat{f}(\xi) \in C^0(\mathbb{R}^n)$.

For any $\xi, \xi' \in \mathbb{R}^n$,

$$\widehat{f}(\xi) - \widehat{f}(\xi') = \int_{\mathbb{R}^n} f(x)\Big(e^{-2\pi i x\cdot\xi} - e^{-2\pi i x\cdot\xi'}\Big)\, dx.$$

Then

$$|\widehat{f}(\xi) - \widehat{f}(\xi')| \le \int_{\mathbb{R}^n} \underbrace{|f(x)||e^{-2\pi i x\cdot\xi} - e^{-2\pi i x\cdot\xi'}|}_{g(x,\xi)}\, dx \tag{1.2}$$

We observe that $0 \le g(x,\xi) \le 2|f(x)| \in L^1(\mathbb{R}^n)$ independently of x and $g(x,\xi) \to 0$ as $\xi \to \xi'$. Then Lebesgue's dominated convergence theorem implies that the right hand side of (1.2) tends to zero as $\xi \to \xi'$. This gives us that $\widehat{f}(\xi) \in C^0(\mathbb{R}^n)$. □

To complete the proof of Proposition 1.16 we use the following result known as the Riemann-Lebesgue lemma. More precisely,

Lemma 1.17 (Riemann-Lebesgue)**.** *If* $f \in L^1(\mathbb{R}^n)$*, then* $\widehat{f}(\xi) \to 0$*, as* $|\xi| \to \infty$*.*

Proof. **Case** $\mathbf{n} = 1$**.**

Let $g(x) = \sum_{j=1}^{m} \alpha_j \chi_{(a_j,b_j)}(x)$ be a step function. Then

$$\begin{aligned}\widehat{g}(\xi) &= \sum_{j=1}^{m} \alpha_j \int_{a_j}^{b_j} e^{-2\pi i x\xi}\, dx \\ &= \sum_{j=1}^{m} \alpha_j \frac{1}{-2\pi i\xi}\Big(e^{-2\pi i b_j\xi} - e^{-2\pi i a_j \xi}\Big) \quad \text{whenever } \xi \neq 0.\end{aligned}$$

Then

$$|\widehat{g}(\xi)| \leq \big(\sum_{j=1}^{m} |\alpha_j|\big)\frac{1}{2\pi|\xi|} \to 0 \quad \text{as } |\xi| \to \infty.$$

Since the step functions are dense in $L^1(\mathbb{R})$, given $f \in L^1(\mathbb{R})$ and $\epsilon > 0$, there exists g a step function such that $\|f - g\|_{L^1} < \epsilon$.

Hence

$$\begin{aligned}|\widehat{f}(\xi)| &\leq |(\widehat{f} - \widehat{g})(\xi)| + |\widehat{g}(\xi)| \\ &\leq \epsilon + |\widehat{g}(\xi)|.\end{aligned}$$

Letting $|\xi| \to \infty$ in the inequality above yields

$$\limsup_{|\xi|\to\infty} |\widehat{f}(\xi)| \leq \epsilon, \epsilon > 0.$$

We conclude that

$$|\widehat{f}(\xi)| \to 0 \quad \text{as } |\xi| \to \infty.$$

Case n > 1**.** Exercise. □

Basically, there are three types of linear operators defined on functions spaces.

(i) Integral operators

$$T_\kappa f(x) = \int_\Omega \kappa(x,y)\, f(y)\, dy.$$

$$\begin{aligned}&T_\kappa : L^p(\Omega) \to L^p(\Omega) \\ &\kappa : \Omega \times \Omega \to \mathbb{C}, \quad \Omega \subseteq \mathbb{R}^n.\end{aligned}$$

κ is called the kernel of the operator.

(ii) Multiplication operators.

$$\begin{aligned}&F : \Omega \subseteq \mathbb{R}^n \to \mathbb{C} \\ &T_F f(x) = F(x) f(x), \\ &T_F : L^p(\Omega) \to L^p(\Omega).\end{aligned}$$

(iii) Differential operators.
For example:

$$Tf = -\Delta f = -(\partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_n}^2)f.$$

These operators are typically unbounded operators.

1.1. **Examples of Integral Operators.**

Example 1.18 (Hilbert-Schmidt Operators).
Let $\kappa \in L^2(\Omega \times \Omega)$ where $\Omega \subseteq \mathbb{R}^n$ is a open set.
Define

$$\begin{aligned} T_\kappa &: L^2(\Omega) \to L^2(\Omega) \\ f &\mapsto T_\kappa f : x \in \Omega \mapsto \int_\Omega \kappa(x,y) f(y)\, dy. \end{aligned}$$

We can see that $T_\kappa \in \mathbb{B}(L^2(\Omega))$.

In fact, using the definition of T_κ and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} |T_\kappa f(x)| = \Big| \int_\Omega \kappa(x,y) f(y)\, dy \Big| &\leq \int_\Omega |\kappa(x,y)||f(y)|\, dy \\ &\leq \Big(\int_\Omega |\kappa(x,y)|^2\, dy \Big)^{1/2} \Big(\int_\Omega |f(y)|^2\, dy \Big)^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} \|T_\kappa f\|_{L^2(\Omega)} &\leq \Big(\int_\Omega \Big(\int_\Omega |\kappa(x,y)|^2\, dy \Big) \|f\|_{L^2(\Omega)}^2\, dx \Big)^{1/2} \\ &= \|\kappa\|_{L^2(\Omega\times\Omega)} \|f\|_{L^2(\Omega)}. \end{aligned}$$

Exercise 1.19.

(i) *Prove that for any $f \in L^2(\Omega)$, $T_\kappa f$ is a measurable function.*
(ii) *Show that $T_\kappa f$ is a compact operator.*

Example 1.20 (Convolution).
Let $f, g : \mathbb{R}^n \to \mathbb{C}$ be mensurable functions, the **convolution** *product of f and g is defined by*

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, dy = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy,$$

whenever the integral makes sense.

Theorem 1.21 (Young)**.**
If $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}.$$

If $f \in L^{p'}(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\|f * g\|_{L^\infty} \le \|f\|_{L^{p'}} \|g\|_{L^p}.$$

Proof. Exercise. □

Remark 1.22. *Using Theorem 1.21 and interpolation theory one can prove a generalized version of Young's inequality. More precisely, for $1 \le p, q \le \infty$ satisfying $\frac{1}{p} + \frac{1}{q} \ge 1$, then $f * g \in L^r(\mathbb{R}^n)$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

and

$$\|f * g\|_{L^r} \le c(n, p, q) \|f\|_{L^p} \|g\|_{L^q}.$$

Example 1.23 (Fourier Transform)**.** *We already defined the Fourier transform for $f \in L^1(\mathbb{R}^n)$, i.e.*

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \dot{\xi}}\, dx, \quad \xi \in \mathbb{R}^n.$$

We proved that $\mathcal{F} \in \mathcal{B}(L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))$ and that if $f \in L^1(\mathbb{R}^n)$, then $f \in C^0_\infty(\mathbb{R}^n)$ (Riemann-Lebesgue lemma).

We see next examples of integrable functions such that $\widehat{f} \notin L^1(\mathbb{R}^n)$.

Consider $f(x) = \chi_{[-1,1]}(x) \in L^1(\mathbb{R}^n)$. The Fourier transform of f is

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{-2\pi i x \cdot \xi}\, dx = \begin{cases} \dfrac{e^{2\pi i x \xi} - e^{-2\pi i x \xi}}{-2\pi i \xi} = \dfrac{\sin(2\pi\xi)}{\pi\xi}, & \text{if } \xi \neq 0, \\ 2, & \text{if } \xi = 0. \end{cases}$$

Thus $\widehat{f}(\xi) = \dfrac{\sin(2\pi\xi)}{\pi\xi}$.

Now we show that $\widehat{f}(\xi) \notin L^1(\mathbb{R})$. For this we observe that for $n \ge 1$ we have that

$$\int_{n\pi}^{(n+1)\pi} |\widehat{f}(\xi)|\, d\xi = \int_{n\pi}^{(n+1)\pi} \frac{|\sin(2\pi\xi)|}{\pi|\xi|}\, d\xi$$
$$\ge \frac{1}{(n+1)\pi} \int_{n\pi}^{(n+1)\pi} |\sin(2\pi\xi)|\, d\xi = \frac{2}{(n+1)\pi}.$$

Adding we obtain

$$I_n = \int_0^{(n+1)\pi} |\widehat{f}(\xi)|\, d\xi = \sum_{k=0}^{n} \int_{k\pi}^{(k+1)\pi} |\widehat{f}(\xi)|\, d\xi$$
$$\geq \frac{2}{\pi} \sum_{k=0}^{n} \frac{1}{(k+1)} = \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k} \to \infty \ \text{as } n \to \infty.$$

Exercise 1.24. *Show that*

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

exists in the sense of a generalized Riemann integral, i.e.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{\substack{\tilde{x}\to\infty \\ y\to-\infty}} \int_{y}^{\tilde{x}} \frac{\sin x}{x}\, dx = \pi.$$

Hint: *Use the residue theorem.*

Remark 1.25. *The Fourier transform spoils the support. We have seen that for $f(x) = \chi_{[-1,1]}(x)$ which a compact support function its Fourier transform $\widehat{f}(\xi) = \dfrac{\sin(2\pi\xi)}{\pi\xi}$ is not a compact support function. We will prove the following general fact,*

Theorem 1.26 (Paley-Wiener)**.** *If $f \in D(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$, then $\widehat{f}$ is an analytic function.*

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