Teoria Espectral

1. More examples of closed, closable, adjoint, self-adjoint operators

In these notes are presented some examples and remarks concerning closed, closable, adjoint, self-adjoint unbounded linear operators.

Example 1.1. *It is easy to construct, using an algebraic basis, a linear operator whose domain is the entire Hilbert space, but which is unbounded. (We are of course assuming that the Hilbert space is infinite dimensional.) By the closed graph theorem, this operator cannot be closed. So it provides an extreme example of an operator which is not closable.*

Example 1.2. *It is also possible for an operator to have many closed extensions. Here is an example. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R})$ and the operator is*

$$\begin{cases} D(A) = \{f \in C_0^\infty(\mathbb{R}) : \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} x\,f(x)\,dx = 0\} \\ \\ A(f)(x) = (1+x^2)f(x). \end{cases}$$

If one takes Fourier transform, this operator becomes the differential operator $-\dfrac{d^2}{d\xi^2} + 1$ with "initial conditions" $\widehat{f}(0) = \dfrac{d\widehat{f}}{d\xi}(0) = 0$.

Set

$$p_0(\xi) = \frac{1}{1+\xi^2}, \qquad p_1(\xi) = \frac{\xi}{1+\xi^2}.$$

Then the closure of A is

$$\begin{cases} D(\overline{A}) = \{f \in L^2(\mathbb{R}) : (1+\xi^2)f \in L^2(\mathbb{R}), (1+\xi^2)f \perp p_0, p_1\} \\ (\overline{A}f)(\xi) = (1+\xi^2)f(\xi). \end{cases}$$

Choose any nonzero $p, q \in \operatorname{span}\{p_0, p_1\}$ and a nonzero $p^\perp \in \operatorname{span}\{p_0, p_1\}^\perp$ which is perpendicular to p. The following are all closed extensions of

A.

$$\begin{cases} D(A_1) = \{f \in L^2(\mathbb{R}) : (1+\xi^2)f \in L^2(\mathbb{R}), (1+\xi^2)f \perp p\} \\ (A_1 f)(\xi) = (1+\xi^2)f(\xi), \end{cases}$$

$$\begin{cases} D(A_2) = \{f \in L^2(\mathbb{R}) : (1+\xi^2)f \in L^2(\mathbb{R})\} \\ (A_2 f)(\xi) = (1+\xi^2)f(\xi), \end{cases}$$

$$\begin{cases} D(A_3) = D(A_1) = \{\alpha \frac{p^\perp}{1+\xi^2} + f : \alpha \in \mathbb{C}, (1+\xi^2)f \in \{p_0, p_1\}^\perp\} \\ A_3(\alpha \frac{p^\perp}{1+\xi^2} + f) = \alpha\, q + (1+\xi^2)f. \end{cases}$$

Example 1.3. *The following example shows that it is possible to have $D(T^*) = \{0\}$. Let*

(i) $\mathcal{H} = L^2(\mathbb{R})$,
(ii) $\{e_n\}_{n\in\mathbb{N}}$ *be an orthonormal basis for* $\mathcal{H}$ *and*
(iii) *for each* $k \in \mathbb{N}$, $f_k(x) = e^{ikx}$. *Note that* $f_k \notin \mathcal{H}$.

We define the domain of T to be $D(T) = \mathcal{B}_0(\mathbb{R})$, the set of all bounded Borel functions on $\mathbb{R}$ that are compact support. This domain is dense in $L^2(\mathbb{R})$. For $\varphi \in D(T)$,

$$T\varphi = \sum_{n=1}^{\infty} \Big[\int_{-\infty}^{\infty} f_n(x)\varphi(x)\, dx \Big] e_n.$$

1. *We first check that T is well-defined. Let $\varphi \in D(T)$. Then there is some integer m such that $\varphi(x)$ vanishes outside of $[-m\pi, m\pi]$. Then, for each $k \in \mathbb{Z}$,*

$$\int_{-\infty}^{\infty} f_k(x)\varphi(x)\, dx = \int_{-m\pi}^{m\pi} f_k(x)\varphi(x)\, dx = \int_{-\pi}^{\pi} e^{ikmt} m\varphi(mt)\, dt$$

is the km-th Fourier coefficient of the function $m\varphi(mt)$. Since the sum of the square of all Fourier coefficients is, up to a factor of 2π, the L^2 norm of $m\varphi(mt)$, which is finite, so T is well-defined.

2. *We now check that $D(T^*) = \{0\}$. Let $\psi \in T^*$ and $\varphi \in D(T) = \mathcal{B}_0(\mathbb{R})$. Choose an $m \in \mathbb{N}$ with $\varphi(x)$ vanishing except for x in $[-m\pi, m\pi]$. Then*

$$\begin{aligned}\langle \varphi, T^*\psi\rangle &= \langle T\varphi, \psi\rangle \\ &= \sum_{n=1}^{\infty}\Big[\int_{-\infty}^{\infty}\overline{f_n(x)\varphi(x)}\,dx\Big]\langle e_n, \psi\rangle \\ &= \sum_{n=1}^{\infty}\langle \varphi, \langle e_n, \psi\rangle \overline{f_n}\rangle.\end{aligned}$$

Since

$$\int_{-m\pi}^{m\pi}\overline{f_k(x)}\,f_l(x)\,dx = \int_{-m\pi}^{m\pi} e^{i(l-k)x}\,dx = \begin{cases} 2m\pi & \text{if } k = l, \\ 0. & \text{if } k \neq l. \end{cases}$$

The series $\sum_{n=1}^{\infty}\langle e_n, \psi\rangle\overline{f_n}$ *converges in* $\mathcal{H}$ *and*

$$\langle \varphi, T^*\psi\rangle = \Big\langle \varphi, \sum_{n=1}^{\infty}\langle e_n, \psi\rangle\overline{f_n}\,\Big\rangle$$

This is true for all bounded Borel functions φ *supported in* $[-m\pi, m\pi]$ *so that*

$$T^*\psi = \sum_{n=1}^{\infty}\langle e_n, \psi\rangle\overline{f_n} \quad \text{a.e. on } [-m\pi, m\pi]$$

and

$$\infty > \|T^*\psi\|_{L^2} \geq \|T^*\psi\|_{L^2([-m\pi,m\pi])} = \sum_{n=1}^{\infty}|\langle e_n, \psi\rangle|^2(2m\pi).$$

Since this is true for all m*, we must have* $\langle e_n, \psi\rangle = 0$ *for all* $n \in \mathbb{N}$ *and hence* $\psi \equiv 0$.

We recall the following notion.

Definition 1.4. *Let* f *be a complex valued function defined on* $[\alpha, \beta]$ *where* $-\infty < \alpha < \beta < \infty$. f *is said* **absolutely continuous** *if there exists an integrable function* g *on* $[\alpha, \beta]$ *such that*

$$f(x) = \int_{\alpha}^{x} g(t)\,dt + f(\alpha).$$

Observe that f *is continuous on* $[\alpha, \beta]$ *and differentiable a.e. with* $f'(x) = g(x)$ *a.e on* $[\alpha, \beta]$*. But it is not necessarily in* $L^2[(\alpha, \beta)]$*. This can be seen by taking the function* $f(x) = x^{1/2}$ *in* $L^2([0, 1])$*. We denote the set of all absolutely continuous functions on* $[\alpha, \beta]$ *by* $AC([\alpha, \beta])$.

Example 1.5. *We consider three different Hilbert spaces:*

$$\begin{aligned} \mathcal{H}_1 &= L^2([\alpha,\beta]), \quad -\infty < \alpha < \beta < \infty, \\ \mathcal{H}_2 &= L^2([\alpha,\infty)), \quad -\infty < \alpha < \infty, \\ \mathcal{H}_3 &= L^2((-\infty,\infty)). \end{aligned}$$

We consider the operators $T_j : D_j \subset \mathcal{H}_j \to \mathcal{H}_j$, $j = 1,2,3$ *defined as*

$$\begin{aligned} D_1 = \{ & g \in \mathcal{H}_1 : g = f \text{ a.e. for } f \in AC([\alpha,\beta]), f(\alpha) = 0 = f(\beta), \text{ and} \\ & f' \in L^2([\alpha,\beta])\} \\ D_2 = \{ & g \in \mathcal{H}_1 : g = f \text{ a.e. for } f \in AC([\alpha,\beta)), \text{for each } \beta > \alpha, f(\alpha) = 0, \\ & \text{and } f' \in L^2([\alpha,\beta])\} \\ D_3 = \{ & g \in \mathcal{H}_1 : g = f \text{ a.e. for } f \in AC([\alpha,\beta]), \text{for each } -\infty < \alpha < \beta < \infty \\ & \text{and } f' \in \mathcal{H}_2\} \end{aligned}$$

with

$$T_j g = f', \quad j = 1,2,3.$$

We will show that

1. *The linear operators* T_j *are unbounded symmetric operators in* $\mathcal{H}_j$, $j = 1,2,3$.
2. T_3 *is a self-adjoint operator.*
3. T_1 *and* T_2 *are not self-adjoint operators.*
4. $T_j = T_j^{**}$, $j = 1,2,3$.

First we notice that $\overline{D_j} = \mathcal{H}_j$, *for each* $j = 1,2,3$. *To show this for* D_1 *we recall that the linear subspace spanned by the set* $\{x^k : k = 0,1,2,\dots\}$ *is dense in* $L^2([\alpha,\beta])$ *because of the class of all complex polynomials is dense in* $L^2([\alpha,\beta])$. *On the other hand,* $x^k \in \overline{D_1}$ *since each* x^k *can be approximated in* $L^2([\alpha,\beta])$ *by a function* f *in* D_1. *For instance, for* $\epsilon > 0$ *suitable, we can take* f *as being*

$$f(x) = \begin{cases} (\alpha+\epsilon)^k \epsilon^{-1}(x-\alpha), & \alpha < x < \alpha+\epsilon, \\ x^k, & \alpha+\epsilon < x < \beta-\epsilon, \\ -(\beta-\epsilon)^k \epsilon^{-1}(x-\beta), & \beta-\epsilon < x < \beta. \end{cases}$$

This means $\overline{D_1} = \mathcal{H}_1$.

To show that $\overline{D_2} = \mathcal{H}_2$ *and* $\overline{D_3} = \mathcal{H}_3$. *We notice that the linear subspace spanned by the set* $\{x^k e^{-x^2/2} : k = 0,1,2,\dots\}$ *is dense in* $L^2((-\infty,\infty))$ *and in consequence their restrictions to* $[\alpha,\infty)$ *are dense in* $L^2([\alpha,\infty))$. *We then can approximate each* $x^k e^{-x^2/2}$ *by a function in* D_2 *or* D_3, *respectively.*

The operators T_j, $j = 1, 2, 3$ are unbounded. We consider for $\alpha < \beta$ and $k \geq \dfrac{2}{\beta - \alpha}$ the functions f_k defined by

$$f_k(x) = \begin{cases} k\,(x-\alpha), & \text{if } \ x \in [\alpha, \alpha + \frac{1}{k}], \\ 2 - k(x-\alpha), & \text{if } \ x \in [\alpha + \frac{1}{k}, \alpha + \frac{2}{k}], \\ 0, & \text{if } \ x \in [\alpha + \frac{2}{k}, \infty). \end{cases}$$

Next we observe that,

$$\|f_k\|^2 = \int_\alpha^{\alpha+2/k} |f_k(x)|^2\, dx = \frac{2}{3k}.$$

and

$$\|if_k'\|^2 = \int_\alpha^{\alpha+2/k} k^2\, dx = 2k.$$

From this we deduce that

$$\frac{\|T_j f_k\|}{\|f_k\|} = \frac{(2k)^{1/2}}{(\frac{2}{3k})^{1/2}} \geq k.$$

Hence the operators are unbounded.

We show next that T_j are symmetric. It is done by using integration by parts. Let $f, g \in D_1$, then

$$\begin{aligned} (T_1 f, g) = (if', g) &= i \int_\alpha^\beta f'(y)\, \overline{g(y)}\, dy \\ &= i\, f(y)\, \overline{g(y)}\Big|_\alpha^\beta - i \int_\alpha^\beta f(y)\, \overline{g(y)}\, dy \\ &= (f, ig') = (f, T_1 g). \end{aligned} \tag{1.1}$$

One can verify the same for the operators T_2 and T_3 by noticing that if $f \in D_2$ then $\lim_{x\to\infty} f(x) = 0$ and for $f \in D_3$ we have that $\lim_{x\to\pm\infty} f(x) = 0$.

Next we will compute de adjoint T_1^* de T_1. Let D_1^* be the set

$$D_1^* = \{g \in \mathcal{H}_1 : g = f \ \text{a.e. where } f \in AC([\alpha, \beta]), f' \in \mathcal{H}_1\}$$

*Notice that (**??**) also holds for $g \in D_1^*$, so the domain of T_1^* contains D_1^* and $T_1^* g = if'$ for $g \in D_1^*$. We shall show that $D(T_1^*) = D_1^*$. This can be done if we prove that $T_1 \subset T_1^*$ but $T_1 \neq T_1^*$. Let $f \in D_1^*$ and let h be the absolutely continuous function given by*

$$h(x) = \int_\alpha^x T_1^* f(s)\, ds + C$$

where C is a constant selected so that

$$\int_\alpha^x [f(s) + ih(s)]\, ds = 0.$$

For every $g \in D_1$, integration by parts yields

$$\begin{aligned}\int_\alpha^\beta ig'(s)\,\overline{f(s)}\,ds = (T_1 g, f) = (g, T_1^* f) = \int_\alpha^\beta g(s)\,\overline{T_1^* f(s)}\,ds \\ = g(s)\,\overline{h(s)}\Big|_\alpha^\beta - \int_\alpha^\beta g'(s)\,\overline{h(s)} = i\int_\alpha^\beta ig'(s)\,\overline{h(s)}.\end{aligned}$$

Then

$$\int_\alpha^\beta g'(s)\,\overline{[f(s) + ih(s)]}\,ds = 0.$$

In particular, taking $g \in D_1$ given by

$$g(x) = \int_\alpha^x [f(s) + ih(s)]\,ds$$

we get that

$$\int_\alpha^\beta |f(s) + ih(s)|^2\,ds = 0.$$

That is,

$$f(x) = -ih(x) = -i\int_\alpha^x T_1^* f(s)\,ds - i\,C, \quad a.e.$$

and h is absolutely continuous with $h'(x) = T_1^* f(x)$. Thus $f \in D_1^*$.

Using the previous analysis we can prove that $T_2^* g = if'$ on the domain

$$D_2^* = \{g \in \mathcal{H}_1 : g = f \ a.e. \ where\ f \in AC([\alpha,\beta]),\ \beta > \alpha,\ f' \in \mathcal{H}_2\}$$

and $T_3^* g = if'$ on the domain $D_3^* = D_3$.

Since $D_1^* \subsetneq D_1$, $D_2^* \subsetneq D_2$ and $D_3^* = D_3$. We have that T_3^* is self-adjoint and that T_1^{**}, T_2^{**} and T_3^{**} are well-defined.

It remains to prove that $T_j = T_j^{**}$ for $j = 1, 2$. In either case we have

$$T_j \subset T_j^{**} \subset T_j^*$$

since $T_j \subset T_j^*$. It suffices then to show that $D_{T_j^{**}} \subset D_j$.

Let $f \in D_{T_j^{**}}$, then for all $g \in D_j^*$ we have

$$(T_j^{**} f, g) = (f, T_j^* g).$$

Since $T_j^{**} \subset T_j^*$, we have $T_j^{**} f = if'$ and then

$$0 = (if', g) - (f, ig').$$

If $j = 1$, this means

$$0 = i\int_\alpha^\beta f'(s)\,\overline{g(s)}\,ds + i\int_\alpha^\beta f(s)\,\overline{g'(s)}\,ds$$
$$= i f(s)\,\overline{g(s)}\Big|_\alpha^\beta = i[f(\beta)\,\overline{g(\beta)} - f(\alpha)\,\overline{g(\alpha)}].$$

Taking first $g(x) = \dfrac{(x-\alpha)}{(\beta-\alpha)} \in D_1^*$ *and then* $g(x) = \dfrac{(\beta-x)}{(\beta-\alpha)} \in D_1^*$, *we obtain* $f(\alpha) = 0 = f(\beta)$ *which implies that* $f \in D_1$.

If $j = 2$, let $g(x) = e^{-(x-\alpha)}$ *to yield* $f(\alpha) = 0$. *Thus* $f \in D_2$.

Remark 1.6. *We observe that T_1 has uncountably many different self-adjoint extensions. Indeed, let $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ and define T_γ in $\mathcal{H}_1$ on*

$$D_{T_\gamma} = \{g \in L^2((\alpha,\beta)) : g = f \text{ a.e. where } f \in AC([\alpha,\beta]), f' \in \mathcal{H}_1, \text{ and } f(\beta) = \gamma\, f(\alpha)\}$$

by $T_\gamma g = i f'$. Each T_γ is self-adjoint and extends T_1. For each γ, we have $T_1 \subset T_\gamma \subset T_1^*$.

Exercise 1.7. *Consider the symmetric unbounded operators T_j, $j = 1,2,3$ defined in Example* **??**. *Use the Basic Criteria to show that*

(a) *T_3 is a self-adjoint operator.*
(b) *T_1 and T_2 are not self-adjoint operators.*

Remark 1.8. *The sprectrum of a linear operator of A is union of the three disjoint following sets:*

(i) *$\sigma_p(A)$ the point spectrum: the set of all eigenvalues.*
(ii) *$\sigma_r(A)$ the residue spectrum: the set of all λ that are not eigenvalues and such that the image of $\lambda - T$ is not dense in X.*
(iii) *$\sigma_c(A)$ the continuous spectrum: the complementary of $\sigma_p(A)$ and $\sigma_r(A)$ it is also the set of λ such that $\lambda - A$ is injective with dense image, but $(\lambda - A)^{-1}$ is not continuous.*

Example 1.9. *Here is an example which shows, firstly, that an unbounded operator T may have $\sigma(T) = \emptyset$ and secondly that "just changing the domain of an operator" can change its spectrum. The Hilbert space $\mathcal{H} = L^2([0,1])$.*

(i) *If* $D(T) = AC[0,1]$ *with* $Tf = if'$, *then* $\sigma(T) = \mathbb{C}$.

In fact $\sigma_p(T) = \mathbb{C}$, *since, for any* $\lambda \in \mathbb{C}$, $e^{-i\lambda x}$ *is an eigenfunction for* T *with eigenvalue* λ.

(ii) *If* $D(T) = \{f \in AC[0,1] : f(0) = 0\}$ *with* $Tf = if'$, *then* $\sigma(T) = \emptyset$.

Indeed, for any $\lambda \in \mathbb{C}$, *the resolvent operator* $(\lambda - T)^{-1}$ *is*

$$(R_\lambda(T)\psi)(x) = i \int_0^x e^{-i\lambda(x-t)}\psi(t)\, dt.$$

(iii) *Let* $\alpha \in \mathbb{C}$ *be nonzero. If* $D(T) = \{f \in AC[0,1] : f(0) = \alpha f(1)\}$ *with* $Tf = if'$, *then* $\sigma(T) = \{-i \ln \alpha + 2k\pi : k \in \mathbb{Z}\}$. *Again the spectrum consists solely of eigenvalues. If* $\lambda = -i \ln \alpha + 2k\pi$ *for some* $k \in \mathbb{Z}$, *then* $e^{i\lambda x}$ *is an eigenvalue for* T *with eigenvalue* λ. *For* λ *not of the form* $-i \ln \alpha + 2k\pi$ *for all* $k \in \mathbb{Z}$, *the resolvent operator* $(\lambda - T)^{-1}$ *is*

$$(R_\lambda(T)\psi)(x) = i \int_0^x G_\lambda(x,t)\psi(t)\, dt$$

with

$$G_\lambda(x,t) = \begin{cases} \dfrac{i\lambda\, e^{i\lambda(t-x-1)}}{1 - \alpha e^{-i\lambda}}, & \text{if } x < t, \\[2ex] \dfrac{i\, e^{i\lambda(t-x)}}{1 - \alpha e^{-i\lambda}}, & \text{if } x > t. \end{cases}$$