Teoria Espectral

The idea of these notes is to complement Section 1.3 in the notes regarding Fourier Transform. The material presented here was taking from the book of Jeffrey Rauch, *Partial Differential Equations, Graduate Texts in Mathematics, Springer-Verlag* (see [1]).

1. Distributions

The distribution theory arises in several contexts. One is the treatment of impulsive forces. Newton's second law affirms that the rate of change of momentum is equal to the force applied, $\dfrac{dp}{dt} = F$. Consider an intense force which acts over a very short interval of time $t_0 < t < t_0 + \Delta t$. An example is the force applied by the strike of a hammer. The impulse, I, is defined as $I := \int F(t)\, dt$ thus

$$p(t_0 + \Delta t) = p(t_0) + I.$$

In the limit, as Δt tends to zero, one arrives to an idealized force which acts instantaneously to produce a jump I in the momentum p. Formally, the force law satisfies

$$F = 0 \quad \text{for } t \neq 0 \quad \text{and} \quad \int F(t)\, dt = I. \tag{1.1}$$

This idealized force is denoted $I\delta_{t_0}$, and δ_{t_0} is called *Dirac's delta function* though no function can satisfy (1.1). The idealized equation of motion is $\dfrac{dp}{dt} = \delta_{t_0}$. The solution satisfies $p(t+) - p(t-) = I$. Such idealizations have shown to be useful in a variety of problems of mechanics and electricity.

The mathematical framework was developed by Lawrence Schwartz in the 1940's.

We introduce some notation next. Let $\Omega \subset \mathbb{R}^n$ an open subset. The set of all infinitely differentiable functions with compact support $C_0^\infty(\Omega)$ will be denoted by $\mathcal{D}(\Omega)$ and $C^\infty(\Omega)$ the set of all infinitely differentiable functions on Ω will be denoted by $\mathcal{E}(\Omega)$. These sets of functions are referred as test functions.

Definition 1.1. *A distribution on an open $\Omega \subset \mathbb{R}^n$ is a linear map $l : \mathcal{D}(\Omega) \to \mathbb{C}$, which is continuous in the sense that $\{\varphi_n\} \subset \mathcal{D}(\Omega)$ satisfies*

(i) *there is a compact $K \subset \Omega$ such that for all n, supp$(\varphi_n) \subset K$ and*

(ii) *there is a $\varphi \in \mathcal{D}(\Omega)$ such that for all $\alpha \in \mathbb{N}^n$, $\partial^\alpha \varphi_n$ converges uniformly to $\partial^\alpha \varphi$,*

then $l(\varphi_n) \to l(\varphi)$. The set of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$. When φ_n, φ satisfy (i) and (ii) we say that φ_n converges to φ in $\mathcal{D}(\Omega)$.

The action of a distribution $l \in \mathcal{D}'(\Omega)$ on a test function $\varphi \in \mathcal{D}(\omega)$ is usually denoted $\langle f, \varphi \rangle$. The set $\mathcal{D}'(\Omega)$ is a complex vector space.

Example 1.2. *If $f \in L^1_{\text{loc}}(\Omega)$, then there is a natural distribution l_f defined by*

$$\langle l, \varphi \rangle = \int f(x)\varphi(x)\, dx.$$

In this sense, the distributions are generalizations of functions and are sometimes called **generalized functions**. *Two locally integrable functions define the same distribution if and only if the functions are equal almost everywhere. We say that a distribution l is a locally integrable function and write $l \in L^1_{\text{loc}}(\Omega)$ if $l = l_f$ for a $f \in L^1_{\text{loc}}(\Omega)$. Similarly, we say that l is continuous (resp. $C^\infty(\Omega)$) if $l = l_f$, for a $f \in C(\Omega)$ (resp. $C^\infty(\Omega)$).*

Example 1.3. *If $x_0 \in \Omega$, then $\langle l, \varphi \rangle \equiv \varphi(x_0)$ is a distribution denoted δ_{x_0} and called the Dirac delta at x_0. When x_0 is not mentioned it is assumed to be the origin. More generally, $\langle l, \varphi \rangle \equiv \partial^\alpha \varphi(x_0)$ is a distribution.*

The following proposition characterizes the distributions. More precisely.

Proposition 1.4. *A linear map $l : \mathcal{D}(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$ if and only if for every compact subset $K \subset \Omega$ there is an integer $N(K, l)$ and a constant $c \in \mathbb{R}$ such that for all $\varphi \in \mathcal{D}(\Omega)$ with support in K*

$$|\langle l, \varphi \rangle| \le c \|\varphi\|_{C^N}, \qquad \|\varphi\|_{C^N} = \sum_{|\alpha| \le N} \max |\partial^\alpha \varphi|. \tag{1.2}$$

Proof. If $l \in \mathcal{D}'(\Omega)$ then is clear that (1.2) holds.

Suppose now that (1.2) does not hold for a compact K. For each integer n, choose $\varphi_n \in \mathcal{D}(\Omega)$ with support in K such that

$$|\langle l, \varphi \rangle| > 1 \quad \text{and,} \quad \|\varphi\|_{C^N} < \frac{1}{n}. \tag{1.3}$$

Then φ_n satisfy (i) and (ii) with $\varphi = 0$, but $\langle l, \varphi \rangle$ does not converge to zero thus l is not a distribution. □

Definition 1.5. *A sequence of distributions $l_n \in \mathcal{D}'(\Omega)$ converges to $l \in \mathcal{D}'(\Omega)$ if and only if for every test function $\varphi \in \mathcal{D}(\Omega)$, $l_n(\varphi) \to l(\varphi)$. This convergence is denoted $l_n \rightharpoonup l$ or $l_n \overset{\mathcal{D}'}{\to} l$.*

Example 1.6. *If $j \in \mathcal{D}(\Omega)$ with $\int j(x)\, dx = 1$, let $j_\epsilon(x) = \epsilon^{-n} j(x/\epsilon)$. Then $j_\epsilon \to \delta_0$.*

1.1. **Operations with distributions.** The great utility of distributions lies on the fact that the standard operations of calculus extend to $\mathcal{D}'(\Omega)$. For instance, one can differentiate distributions. This is quite important in the study of differential equations.

The recipe for defining operations on distributions is basically the same: pass the operator onto the test function.

Example 1.7. *We recall the translation operator* $\tau_y f = f(x-y)$, $y \in \mathbb{R}^n$. *Let* $l \in \mathcal{D}'(\mathbb{R}^n)$*, the translate of* l *by the vector* y, $\tau_y l$*, is defined as follows. If* l *were equal to the function* f*, then*

$$\langle \tau_y I, \varphi \rangle = \int f(x-y)\varphi(x)\,dx = \int f(z)\varphi(z+y)\,dz = \langle l, \tau_{-y}\varphi \rangle, \;\; \varphi \in \mathcal{D}(\mathbb{R}^n).$$

This motivates the definition, $\langle \tau_h l, \varphi \rangle = \langle l, \tau_{-h}\varphi \rangle$*. It is easy to check that* $\tau_h l$ *defined as above is a distribution and that definition agrees with* $\tau_h f$ *when* $l = l_f$*.*

Example 1.8. *To differentiate a distribution* l *on* $\mathbb{R}^n$*, we form the difference quotients which could converge to* $\dfrac{\partial l}{\partial x_j}$*. Let* e_j *be the vector with* j*th coordinate equals* 1 *and* 0 *in the others. The difference quotients are given by*

$$\Big\langle \frac{\tau_{-he_j} l - l}{h}, \varphi \Big\rangle \equiv \Big\langle l, \frac{\tau_{he_j}\varphi - \varphi}{h} \Big\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1.4}$$

The test functions on the right converge to $-\dfrac{\partial \varphi}{\partial x_j}$*, so the continuity of* l *implies that the right hand side of* (1.4) *converges to* $\langle l, -\dfrac{\partial \varphi}{\partial x_j} \rangle$*. This suggests that*

$$\Big\langle \frac{\partial l}{\partial x_j}, \varphi \Big\rangle \equiv \langle l, -\frac{\partial \varphi}{\partial x_j} \rangle. \tag{1.5}$$

This defines a distribution and if $f \in C^1(\Omega)$ *and* $l = l_f$*, the derivatives of* l *are equal to the distributions* $l_{\frac{\partial f}{\partial x_j}}$*. Thus the operator* ∂/∂_{x_j} *on* $\mathcal{D}'$ *is an extension of* ∂/∂_{x_j} *on* $\mathcal{D}$*.*

Let us apply the above procedure to find the derivative in distributions sense of the Heaviside function $H(x) = \chi_{[0,\infty)}(x)$ *defined on* $\mathbb{R}$*. The difference quotient*

$$\frac{\tau_{-h}H - H}{h} = h^{-1}\chi_{[0,h]}$$

converges to δ *in the sense of distributions. Thus* $\frac{dH}{dx} = \delta$*. Observe that the difference quotient converge to zero almost everywhere. Since* H *is not constant, zero should not be the desired derivative. The pointwise limit gives the wrong answer and the distribution derivative is the right answer.*

The operations on distributions discussed so far are particular cases of a general algorithm.

Proposition 1.9 (P.D. Lax)**.** *Suppose that L is a linear map from $\mathcal{D}(\Omega_1)$ to $\mathcal{D}(\Omega_2)$, which is sequentially continuous in the sense that $\varphi_n \to \varphi$ implies $L(\varphi_n) \to L(\varphi)$. Suppose, in addition, that there is an operator L', sequentially continuous from $\mathcal{D}(\Omega_2)$ to $\mathcal{D}(\Omega_1)$, which is the transpose of l in the sense that*

$$\langle L(\varphi), \psi \rangle = \langle \varphi, L'(\psi) \rangle \quad \text{for all } \varphi \in \mathcal{D}(\Omega_1), \psi \in \mathcal{D}(\Omega_2).$$

Then the operator L extends to a sequentially continuous map of $\mathcal{D}'(\Omega_2)$ to $\mathcal{D}'(\Omega_1)$ given by

$$\langle L(l), \psi \rangle = \langle l, L'(\psi) \rangle \quad \text{for all } l \in \mathcal{D}'(\Omega_1), \text{ and } \psi \in \mathcal{D}(\Omega_2). \tag{1.6}$$

Proof. The sequential continuity of L' shows that $L(l)$ defined in (1.6) is a distribution. If $l = l_\varphi$ for some $\varphi \in D(\Omega_1)$, then

$$\langle L(l), \psi \rangle \equiv \langle l, L'(\psi) \rangle = \int_{\Omega_1} \varphi(x) L'(\psi)(x)\, dx = \int_{\Omega_2} L(\varphi)(x)\psi(x)\, dx, \tag{1.7}$$

the last equality from the hypothesis that L' is the transpose of L. Thus $L(l)$ is the distribution associated to $L(\varphi)$ which proves that L defined by (1.6) extends $L\big|_{\mathcal{D}'}$.

Finally, if $l_n \rightharpoonup l$ in $\mathcal{D}'(\Omega_1)$, it follows immediately from (1.6) that $L(l_n) \rightharpoonup L(l)$ proving the sequentially continuity of L.

□

Remark 1.10. *The proof of the uniqueness of this extension can be seen in* [1] *Appendix Proposition 8.*

Example 1.11. *If $a(x) \in C^\infty(\Omega)$, $(\equiv \mathcal{E}(\Omega))$, then the map $L(\varphi) \equiv a\varphi$ is equal to its own transpose. That is,*

$$\langle L(\varphi), \psi \rangle = \int (a(x)\varphi(x))(\psi)\, dx = \int (\varphi(x))(a(x)\psi)\, dx = \langle \varphi, L(\psi) \rangle.$$

Thus for $l \in \mathcal{D}'(\Omega)$, al is a well-defined distribution given by $\langle al, \varphi \rangle \equiv \langle l, a\varphi \rangle$.

Example 1.12. *If $\Omega_2 = y + \Omega$ and L_τ is translation by y, then $L' = \tau_{-y}$ is sequentially continuous. Therefore for $l \in \mathcal{D}'(\Omega_1)$ the translates of l are well defined by $\langle \tau_y l, \varphi \rangle \equiv \langle l,\ \tau_{-y}\varphi \rangle$. The reflection operator $\widetilde{\varphi}(x) = \varphi(-x)$ is its own transpose, thus $\tilde{l}$ is a well-defined distribution on the reflection of Ω.*

Example 1.13. *If $L = \partial^\alpha$ ($\partial^\alpha \equiv \partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n}$, $\alpha \in \mathbb{Z}^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$). Integration by parts gives $L' = (-1)^{|\alpha|}\partial^\alpha$ which is sequentially continuous on $\mathcal{D}$. Thus the derivatives of distributions are defined by*

$$\langle \partial^\alpha l, \varphi \rangle \equiv \langle l, (-1)^{|\alpha|}\partial^\alpha \varphi \rangle.$$

Once we have defined multiplication and derivatives we can compute a product rule as being

$$\langle \partial(al), \psi\rangle \equiv \langle l, -a\partial\psi\rangle = \langle l, -\partial(a\psi)\rangle + \langle l, (\partial a)\psi\rangle = \langle a\partial l + (\partial a)l, \psi\rangle.$$

Following this procedure inductively we can define the usual Leibniz for $\partial^\alpha(al)$.

If $P(x, D) = \sum a_\alpha(x)\partial^\alpha$ is a linear partial differential operator with coefficients in $\mathcal{E}(\Omega)$, then P maps $\mathcal{D}'(\Omega)$ to itself with $\langle Pl, \varphi' \equiv \langle l, P'\varphi\rangle$ where the transpose of P is given by

$$P'\psi = \sum(-1)^{|\alpha|}\partial^\alpha(a_\alpha\psi).$$

1.2. **Convolution.** Suppose that $\Omega = \mathbb{R}^n$ and $\varphi \in \mathcal{D}'(\mathbb{R}^n)$. Ler L be the operator $L(\psi) = \varphi * \psi$. The Leibniz rule for differentiating under the integral implies that L maps $\mathcal{D}'(\mathbb{R}^n)$ continuously to itself. The Fubini theorem shows that the transpose of L is convolution with $\widetilde{\varphi}$. Thus $\varphi * l$ makes sense for any $l \in \mathcal{D}'(\mathbb{R}^n)$ and it is given by

$$\langle \varphi * l, \psi\rangle \equiv \langle l, \widetilde{\varphi}\psi\rangle.$$

Example 1.14. *We compute* $\varphi * \delta$

$$\langle \varphi * \delta, \psi\rangle \equiv \langle \delta, \widetilde{\varphi}\psi\rangle = (\widetilde{\varphi} * \psi)(0) = \int \varphi(y)\psi(y)\, dy = \langle \varphi, \psi\rangle.$$

Therefore $\varphi * \delta = \varphi$.

It is not difficult to show that for $l \in \mathcal{D}'(\mathbb{R}^n)$,

$$\partial^\alpha(\varphi * l) = \varphi * \partial^\alpha l = (\partial^\alpha\varphi) * l.$$

We end this section with the following result whose proof can be see in [1] Appendix Proposition 3.

Proposition 1.15. *If* $l \in \mathcal{D}'(\mathbb{R}^n)$ *and* $\varphi \in \mathcal{D}(\mathbb{R}^n)$, *then* $l * \varphi$ *is equal to the* C^∞ *function whose value at* x *is* $\langle l, \tau_x(\widetilde{\varphi})\rangle$.

1.3. **Tempered distributions.** Recall that $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, the space of the C^∞ functions decaying at infinity, that is,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty : \|\varphi\|_{\alpha,\beta} \equiv \|x^\alpha\partial^\beta\varphi\|_{L^\infty(\mathbb{R}^n)} < \infty, \text{ for any } \alpha, \beta \in (\mathbb{Z}^+)^n\}.$$

Definition 1.16. *A* **tempered distribution** *is a continuous linear functional on* $\mathcal{S}(\mathbb{R}^n)$. *The set of all tempered distribution is denoted by* $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 1.17. *A linear map* $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ *is continuous if and only if there exist* $N \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that for all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$|\langle T, \varphi\rangle| \leq c \sum_{|\alpha|\leq N,\, |\beta|\leq N} \|x^\beta\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^n)}. \tag{1.8}$$

Corollary 1.18. *A distribution* $T \in \mathcal{S}'(\mathbb{R}^n)$ *extends uniquely to an element of* $\mathcal{S}'(\mathbb{R}^n)$ *if and only if there exist* $N \in \mathbb{N}$ *and* $c \in \mathbb{R}$ *such that* (1.8) *holds for all* $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

In particular, we have

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n).$$

Example 1.19. *If* f *is a Lebesgue measurable function on* $\mathbb{R}^n$ *such that for some* M, $(1+|x|^2)^{-M} f \in L^1(\mathbb{R}^n)$, *then the distribution defined by* f *is tempered since*

$$\begin{aligned}\langle f, \varphi\rangle &= \langle (1+|x|^2)^{-M} f, (1+|x|^2)^M \varphi\rangle \\ &\le \|(1+|x|^2)^{-M} f\|_{L^1} \|(1+|x|^2)^M \varphi\|_{L^\infty} \le c_{f,M} \| \varphi \|_{2M,0}.\end{aligned}$$

Example 1.20. *If* $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, *then* $f \in \mathcal{S}'$ *since these functions satisfy the condition of Example 1.19, if one chooses* M *so large that* $(1+|x|^2)^{-M} \in L^q(\mathbb{R}^n)$ *and then uses Hölder's inequality.*

Definition 1.21. *A sequence* $T_n \in \mathcal{S}'(|r^n)$ *converges to* $\mathcal{S}'(\mathbb{R}^n)$ *if and only if for all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle T_n, \varphi\rangle \to \langle T, \varphi\rangle \qquad \text{as } n \to \infty.$$

We write $T_n \overset{\mathcal{S}'}{\to} T$.

In the next we mainly interested in extending to $\mathcal{S}'$ the basic linear operators of analysis, for instance ∂^α and $\mathcal{F}$.

Given a continuous linear operator $L : \mathcal{S} \to \mathcal{S}$, the transpose L' maps $\mathcal{S}' \to \mathcal{S}'$. For $T \in \mathcal{S}'(\mathbb{R}^n)$, $L'T \in \mathcal{S}'$ is defined by

$$\langle L'T, \varphi\rangle \equiv \langle T, L\varphi\rangle \quad \text{for all} \quad \varphi \in \mathcal{S}. \tag{1.9}$$

The next proposition shows that the identity can sometimes be used to extend L.

Proposition 1.22. *Suppose that* $L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *is a continuous linear map and that the restriction of the transpose operator to* $\mathcal{S}$, $L'\big|_{\mathcal{S}}$, *is a continuous map of* $\mathcal{S}$ *to itself. Then* L *has a unique sequentially continuous extension to a linear map* $L : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *defined by*

$$\langle LT, \varphi\rangle \equiv \langle T, L'\varphi\rangle, \qquad \text{for all} \quad T \in \mathcal{S}', \ \varphi \in \mathcal{S}.$$

Proof. See Proposition 4 page 77 in [1]. □

Remark 1.23. *This proposition identifies when passing the operator to the test function yields a good extension.*

For a general L, *one will not even have* $L'\varphi \in \mathcal{S}$ *for* $\varphi \in \mathcal{S}$. *The hypothesis on* L' *is very restrictive. However, the translation, the dilation, multiplication by a convenient function* M *(see Exercise 1.28 below), differentiation* ∂^α *and Fourier transform* $\mathcal{F}$ *are operators which are included in this proposition.*

For $T \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, we have

$$\langle (\partial^\alpha)' T, \varphi \rangle \equiv \langle T, (\partial^\alpha)\varphi \rangle.$$

If $T \in \mathcal{S}$, the right-hand side is equal to

$$\int T(x)\partial_x^\alpha \varphi(x)\, dx = \int (-\partial_x)^\alpha T(x)\varphi(x)\, dx = \langle (-\partial_x)^\alpha T, \varphi \rangle.$$

Thus, for such T, $(\partial^\alpha)' T = (-\partial)^\alpha T$.

Similarly, for $T \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$,

$$\langle \mathcal{F} T, \varphi \rangle \equiv \langle T, \mathcal{F}\varphi \rangle.$$

For $T \in \mathcal{S}$, the duality identity

$$\langle \mathcal{F}\varphi, \psi \rangle = \langle \varphi, \mathcal{F}\psi \rangle, \quad \text{for all} \quad \varphi, \psi \in \mathcal{S},$$

shows that this is equal to $\langle \mathcal{F}T, \varphi \rangle$, whence $\mathcal{F}'\big|_{\mathcal{S}} = \mathcal{F}$.

1.4. **Applications.** First consider the solvability of the equation

$$(1 - \Delta) u = f \tag{1.10}$$

For u, f in $\mathcal{S}'$ this is equivalent to

$$(1 + |\xi|^2)\widehat{u} = \mathcal{F} f,$$

hence

$$\widehat{u} = (1 + |\xi|^2)^{-1} \mathcal{F} f. \tag{1.11}$$

Proposition 1.24. *For any $f \in \mathcal{S}'(\mathbb{R}^n)$ there exists exactly one solution $u \in \mathcal{S}'(\mathbb{R}^n)$ to* (1.10)*. The solution is given by formula* (1.11)*. In particular, if $f \in \mathcal{S}$, then $u \in \mathcal{S}$. If $f \in L^2$, then for all $|\alpha| \leq 2$, $D^\alpha u \in L^2(\mathbb{R}^n)$.*

The second application is a Liouville-type theorem. More precisely.

Theorem 1.25 (Generalized Liouville Theorem)**.** *Suppose that $P(D)$ is a constant coefficient partial differential operator such that $P(\xi) \neq 0$ for $\xi \neq 0$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies $Pu = 0$, then u is a polynomial in x.*

Proof. Taking Fourier transform of the equation we obtain

$$\mathcal{F}(P(D))u = P(\xi)\widehat{u} = 0.$$

Since $P(\xi) \neq 0$ if $\xi \neq 0$ it follows that $supp\, \widehat{u} \subset \{0\}$.

Thus $\mathcal{F}u$ has to be a finite linear combination of derivatives of the delta function

$$\widehat{u} = \sum c_\alpha D^\alpha \delta.$$

Applying the inverse Fourier transform we get

$$u = \sum c_\alpha \mathcal{F} D^\alpha \delta = \sum c_\alpha (-x)^\alpha \mathcal{F}\delta = \sum c_\alpha (-x)^\alpha (2\pi)^{-n/2},$$

a polynomial in x. □

Corollary 1.26. *The only bounded harmonic (resp. holomorphic) functions on $\mathbb{R}^n$ (resp. $\mathbb{C}$) are the constants.*

Next we consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t). \tag{1.12}$$

Both sides of the equation make sense if u is a distribution. If the equality holds we say that u is weak solution. Recall that u is said a classical solution if $u \in C^2(\mathbb{R}^2)$ and the identity is satisfied.

Consider a traveling wave $u(x,t) = f(x-t)$, $f \in L^1_{\text{loc}}(\mathbb{R})$. It is clear that $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ and so it defines a distribution. Is it a weak solution?

Using the differentiation operator definition we find that

$$\Big\langle \frac{\partial^2}{\partial t^2} u, \varphi \Big\rangle = \Big\langle u, \frac{\partial^2}{\partial t^2}\varphi \Big\rangle = \iint f(x-t) \frac{\partial^2}{\partial t^2}\varphi(x,t)\, dxdt$$

$$\Big\langle \frac{\partial^2}{\partial x^2} u, \varphi \Big\rangle = \Big\langle u, \frac{\partial^2}{\partial x^2}\varphi \Big\rangle = \iint f(x-t) \frac{\partial^2}{\partial x^2}\varphi(x,t)\, dxdt.$$

Hence

$$\Big\langle \frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u, \varphi \Big\rangle = \iint f(x-t) \Big(\frac{\partial^2}{\partial t^2}\varphi - \frac{\partial^2}{\partial x^2}\varphi \Big)(x,t)\, dxdt.$$

We would to like to show that this is zero. To do so, we make the change of variables $y = x - t$, $z = x + t$, $dxdt = \frac{1}{2}\, dydz$. and

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -4 \frac{\partial}{\partial y}\frac{\partial}{\partial z}.$$

Thus,

$$\iint f(x-t) \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \Big) \varphi(x,t)\, dxdt = -2 \iint f(y) \frac{\partial^2 \varphi}{\partial y \partial z}(y,z)\, dzdy$$

We claim that integration in z yields zero. Indeed, we observe that

$$\int_a^b \frac{\partial^2 \varphi}{\partial y \partial z}(y,z)\, dz = \frac{\partial \varphi}{\partial y}(y,b) - \frac{\partial \varphi}{\partial y}(y,a).$$

Thus

$$\int_a^b \frac{\partial^2 \varphi}{\partial y \partial z}(y,z)\, dz = 0$$

since φ and $\frac{\partial \varphi}{\partial y}$ vanish in a bounded set. Therefore $u(x,t) = f(x-t)$ is a weak solution of (1.12).

Next we investigate whether $\log(x^2 + y^2)$ is a weak solution of the Laplace equation

$$\Delta u(x,y) = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big) u(x,y) = 0. \tag{1.13}$$

We have to check that

$$\Big\langle \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big) u, \varphi \Big\rangle = \Big\langle u, \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \Big) \varphi \Big\rangle = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^2).$$

Employing polar coordinates (r,θ) we have that

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \text{and } dxdy = r\,drd\theta.$$

Then for $u(x,y)=\log(x^2+y^2)$ we would like to know whether

$$\int_0^{2\pi}\int_0^{\infty}\log r^2\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big)\varphi(r,\theta)\,rdrd\theta=0$$

is true. To avoid the singularity of u at the origin we will integrate in r in (ϵ,∞) and then we make ϵ tends to 0.

We first note

$$\int_0^{2\pi}\log r^2\,\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\varphi(r,\theta)\,r\theta=\frac{1}{r}\log r^2\,\frac{\partial\varphi}{\partial\theta}(r,\theta)\Big|_0^{2\pi}=0 \tag{1.14}$$

since $\dfrac{\partial\varphi}{\partial\theta}$ is periodic. Therefore this term is always 0.

On the other hand we have

$$\int_\epsilon^\infty \log r^2\frac{\partial}{\partial r}\varphi(r,\theta)\,dr=-\int_\epsilon^\infty\frac{\partial}{\partial r}(\log r^2)\varphi(r,\theta)\,dr-\log(\epsilon^2)\varphi(\epsilon,\theta). \tag{1.15}$$

and

$$\begin{aligned}\int_\epsilon^\infty r\log r^2\frac{\partial^2}{\partial r^2}\varphi(r,\theta)\,dr &= -\int_\epsilon^\infty\frac{\partial}{\partial r}(r\log r^2)\frac{\partial}{\partial r}\varphi(r,\theta)\,dr\\ &\quad -\epsilon\log(\epsilon^2)\frac{\partial}{\partial r}\varphi(\epsilon,\theta)\\ &= \int_\epsilon^\infty\frac{\partial^2}{\partial r^2}(r\log r^2)\varphi(r,\theta)\,dr\\ &\quad -\epsilon\log(\epsilon^2)\frac{\partial}{\partial r}\varphi(\epsilon,\theta)\\ &\quad +\frac{\partial}{\partial r}(r\log r^2)\Big|_\epsilon\varphi(\epsilon,\theta).\end{aligned} \tag{1.16}$$

Now $\dfrac{\partial}{\partial r}(r\log r^2)=\log r^2+2$, $\dfrac{\partial^2}{\partial r^2}(r\log r^2)=\dfrac{2}{r}$ and $\dfrac{\partial}{\partial r}(\log r^2)=\dfrac{2}{r}$

Gathering together the information in (1.14), (1.15) and (1.16) we obtain

$$\begin{aligned}\int_0^{2\pi}\int_\epsilon^\infty\log r^2\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\Big)\varphi(r,\theta)\,rdrd\theta&\\ =\int_0^{2\pi}\int_\epsilon^\infty\Big(-\frac{2}{r}+\frac{2}{r}\Big)\varphi(r,\theta)\,drd\theta&\\ +\int_0^{2\pi}(-\log\epsilon^2+\log\epsilon^2+2)\varphi(\epsilon,\theta)\,d\theta&\\ +\int_0^{2\pi}(-\epsilon\log\epsilon^2)\frac{\partial\varphi}{\partial r}(\epsilon,\theta)\,d\theta.&\end{aligned}$$

Thus

$$\langle \Delta u, \varphi\rangle = \lim_{\epsilon\to 0}\ 2\int_0^{2\pi} \varphi(\epsilon,\theta)\,d\theta - \int_0^{2\pi} \epsilon\log\epsilon^2 \frac{\partial\varphi}{\partial r}(\epsilon,\theta)\,d\theta. \tag{1.17}$$

Since φ is continuous $\varphi(\epsilon,\theta)\to\varphi(0,\theta)$ as $\epsilon\to 0$ and so the first term in (1.17) approaches to $4\pi\langle\delta,\varphi\rangle$.

In the second term in (1.17), $\frac{\partial\varphi}{\partial r}(\epsilon,\theta)$ remains bounded while $\epsilon\log\epsilon^2\to 0$ as $\epsilon\to 0$. Hence

$$\Delta\log(x^2+y^2) = 4\pi\delta.$$

Therefore $\log(x^2+y^2)$ is not a weak solution of $\Delta u = 0$.

The previous computations allow us to solve the Poisson equation

$$\Delta u = f \quad \text{for any } f. \tag{1.18}$$

A final remark.

Remark 1.27. *It is clear that $\mathcal{S}'(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$. What is not true is that any distribution in $\mathcal{D}'(\mathbb{R}^n)$ is a tempered distribution. For example the function $f(x)=e^{x^2}$ in $\mathbb{R}$ defines the distribution*

$$\langle f,\varphi\rangle = \int_{-\infty}^{\infty} e^{x^2}\varphi(x)\,dx.$$

Observe that $e^{-x^2/2}\in\mathcal{S}(\mathbb{R})$ and so we have

$$\langle f,\varphi\rangle = \int_{-\infty}^{\infty} e^{x^2}e^{-x^2/2}\,dx = \int_{-\infty}^{\infty} e^{x^2/2} = +\infty$$

which does not define a tempered distribution.

Exercise 1.28. *Prove*

(i) *If $M\in C^\infty(\mathbb{R}^n)$ and $\forall\alpha\in(\mathbb{Z}^+)^n$, there exist N,c such that*

$$|\partial^\alpha M|\le c(1+|x|)^N,$$

then the map $f\to Mf$ is a continuous linear transformation of $\mathcal{S}(\mathbb{R}^n)$ into itself.

(ii) *If in addition, there exist $\gamma,c>0$ such that*

$$|M(x)|\ge c(1+|x|)^{-\gamma},$$

then the mapping is one-to-one and onto with continuous inverse.

Exercise 1.29. *Verify that if f satisfies*

$$\int_{|x|\le A} |f(x)|\,dx \le c\,A^N \quad \text{as} \quad A\to\infty$$

for some constants c and N, then

$$\int_{\mathbb{R}^n} |f(x)\varphi(x)|\,dx < \infty \qquad \forall\varphi\in\mathcal{S}(\mathbb{R}^n).$$

Therefore

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\,dx$$

defines a tempered distribution.

References

[1] J. Rauch, *Partial Differential Equations*, Graduate Texts in Mathematics, 128. Springer-Verlag, New York, 1991. x+263 pp.