Chapter 1
The Fourier Transform

In this chapter, we shall study some basic properties of the Fourier transform. Section 1.1 is concerned with its definition and properties in $L^1(\mathbb{R}^n)$. The case $L^2(\mathbb{R}^n)$ is considered in Section 1.2. The space of tempered distributions is briefly considered in Section 1.3. Finally, Sections 1.4 and 1.5 give an introduction to the study of oscillatory integrals in one dimension and some applications, respectively.

1.1 The Fourier Transform in $L^1(\mathbb{R}^n)$

Definition 1.1. The *Fourier transform* of a function $f \in L^1(\mathbb{R}^n)$, denoted by $\widehat{f}$, is defined as:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i(x\cdot\xi)}dx, \quad \text{for } \xi \in \mathbb{R}^n, \tag{1.1}$$

where $(x \cdot \xi) = x_1\xi_1 + \cdots + x_n\xi_n$.

We list some basic properties of the Fourier transform in $L^1(\mathbb{R}^n)$.

Theorem 1.1. *Let $f \in L^1(\mathbb{R}^n)$. Then:*

1. *$f \mapsto \widehat{f}$ defines a linear transformation from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with*

$$\|\widehat{f}\|_\infty \leq \|f\|_1. \tag{1.2}$$

2. *$\widehat{f}$ is continuous.*
3. *$\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ (Riemann–Lebesgue).*
4. *If $\tau_h f(x) = f(x-h)$ denotes the translation by $h \in \mathbb{R}^n$, then*

$$\widehat{(\tau_h f)}(\xi) = e^{-2\pi i(h\cdot\xi)}\widehat{f}(\xi), \tag{1.3}$$

and

$$(\widehat{e^{-2\pi i(x\cdot h)}f})(\xi) = (\tau_{-h}\widehat{f})(\xi). \tag{1.4}$$

F. Linares, G. Ponce, *Introduction to Nonlinear Dispersive Equations*,
Universitext, DOI 10.1007/978-1-4939-2181-2_1

5. *If* $\delta_a f(x) = f(ax)$ *denotes a dilation by* $a > 0$, *then*

$$\widehat{(\delta_a f)}(\xi) = a^{-n}\widehat{f}(a^{-1}\xi). \tag{1.5}$$

6. *Let* $g \in L^1(\mathbb{R}^n)$ *and* $f * g$ *be the convolution of* f *and* g. *Then,*

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{1.6}$$

7. *Let* $g \in L^1(\mathbb{R}^n)$. *Then,*

$$\int_{\mathbb{R}^n} \widehat{f}(y)g(y)dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)dy. \tag{1.7}$$

Notice that the equality in (1.2) holds for $f \geq 0$, i.e., $\widehat{f}(0) = \|\widehat{f}\|_\infty = \|f\|_1$.

Proof. It is left as an exercise. □

Next, we give some examples to illustrate the properties stated in Theorem 1.1.

Example 1.1 Let $n = 1$ and $f(x) = \chi_{(a,b)}(x)$ (the characteristic function of the interval (a, b)). Then,

$$\begin{aligned}\widehat{f}(\xi) &= \int_a^b e^{-2\pi i x\xi}\, dx \\ &= -\frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{2\pi i\xi} \\ &= -e^{-\pi i(a+b)\xi}\,\frac{\sin(\pi(a-b)\xi)}{\pi\xi}.\end{aligned}$$

Notice that $\widehat{f} \notin L^1(\mathbb{R})$ and that $\widehat{f}(\xi)$ has an analytic extension $\widehat{f}(\xi + i\eta)$ to the whole plane $\xi + i\eta \in \mathbb{C}$. In particular, if $(a, b) = (-k, k)$, $k \in \mathbb{Z}^+$, then we have

$$\widehat{\chi}_{(-k,k)}(\xi) = \frac{\sin(2\pi k\xi)}{\pi\xi}.$$

Example 1.2 Let $n = 1$ and for $k \in \mathbb{Z}^+$ define

$$g_k(x) = \begin{cases} k+1+x, & \text{if } x \in (-k-1, -k+1] \\ 2, & \text{if } x \in (-k+1, k-1) \\ k+1-x, & \text{if } x \in [k-1, k+1) \\ 0, & \text{if } x \notin (-k-1, k+1), \end{cases}$$

i.e., $g_k(x) = \chi_{(-1,1)} * \chi_{(-k,k)}(x)$. The identity (1.6) and the previous example show that

$$\widehat{g_k}(\xi) = \frac{\sin(2\pi\xi)\sin(2\pi k\xi)}{(\pi\xi)^2}.$$

Notice that $\widehat{g_k} \in L^1(\mathbb{R})$ and has an analytic extension to the whole plane $\mathbb{C}$.

Example 1.3 Let $n \geq 1$ and $f(x) = e^{-4\pi^2 t|x|^2}$ with $t > 0$. Then, changing variables $x \to x/\sqrt{t}$ and using (1.5), we can restrict ourselves to the case $t = 1$. From Fubini's theorem we write:

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-4\pi^2|x|^2} e^{-2\pi i(x\cdot\xi)}\, dx &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i \xi_j x_j)}\, dx_j \\ &= \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{(-4\pi^2 x_j^2 - 2\pi i \xi_j x_j + \xi_j^2/4)} e^{-\xi_j^2/4}\, dx_j \\ &= \prod_{j=1}^{n} e^{-\xi_j^2/4} \int_{-\infty}^{\infty} e^{-(2\pi x_j + i\xi_j/2)^2}\, dx_j \\ &= 2^{-n}\pi^{-n/2} e^{-|\xi|^2/4},\end{aligned}$$

where in the last equality, we have employed the following identities from complex integration and calculus:

$$\int_{-\infty}^{\infty} e^{-(2\pi x + i\xi/2)^2}\, dx = \int_{-\infty}^{\infty} e^{-(2\pi x)^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, \frac{dx}{2\pi} = \frac{1}{2\sqrt{\pi}}.$$

Hence,

$$\widehat{e^{-4\pi^2 t|x|^2}}(\xi) = \frac{e^{-|\xi|^2/4t}}{(4\pi t)^{n/2}}. \tag{1.8}$$

Observe that taking $t = 1/4\pi$ and changing variables $t \to 1/16\pi^2 t$ we get:

$$\widehat{e^{-\pi|x|^2}}(\xi) = e^{-\pi|\xi|^2} \quad \text{and} \quad \widehat{\frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}}(\xi) = e^{-4\pi^2 t|\xi|^2},$$

respectively.

Example 1.4 Let $n \geq 1$ and $f(x) = e^{-2\pi|x|}$. Then,

$$\hat{f}(\xi) = \frac{\Gamma[\frac{(n+1)}{2}]}{\pi^{(n+1)/2}} \frac{1}{(1+|\xi|^2)^{(n+1)/2}},$$

where $\Gamma(\cdot)$ denotes the Gamma function. See Exercise 1.1 (i).

Example 1.5 Let $n = 1$ and $f(x) = \dfrac{1}{\pi}\dfrac{1}{1+x^2}$. Using complex integration one obtains the identity:

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+b^2}\, dx = \frac{\pi}{b}\, e^{-ab}, \quad a,\, b > 0.$$

Hence,

$$\frac{1}{\pi}\widehat{\frac{1}{1+x^2}}(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{1+x^2}\,dx$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(2\pi|\xi|x)}{1+x^2}\,dx = e^{-2\pi|\xi|}.$$

One of the most important features of the Fourier transform is its relationship with differentiation. This is described in the following results.

Proposition 1.1. *Suppose* $x_k f \in L^1(\mathbb{R}^n)$, *where* x_k *denotes the kth coordinate of* x. *Then,* $\widehat{f}$ *is differentiable with respect to* ξ_k *and*

$$\frac{\partial \widehat{f}}{\partial \xi_k}(\xi) = (\widehat{-2\pi i x_k f(x)})(\xi). \tag{1.9}$$

In other words, the Fourier transform of the product $x_k f(x)$ is equal to a multiple of the partial derivative of $\widehat{f}(\xi)$ with respect to the kth variable.

To consider the converse result, we need to introduce a definition.

Definition 1.2. Let $1 \le p < \infty$. A function $f \in L^p(\mathbb{R}^n)$ is *differentiable* in $L^p(\mathbb{R}^n)$ with respect to the kth variable, if there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{f(x+he_k)-f(x)}{h} - g(x)\right|^p dx \to 0 \text{ as } h \to 0,$$

where e_k has kth coordinate equals 1 and 0 in the others. If such a function g exists (in this case it is unique), it is called the partial derivative of f with respect to the kth variable in the L^p-norm.

Theorem 1.2. *Let* $f \in L^1(\mathbb{R}^n)$ *and* g *be its partial derivative with respect to the kth variable in the* L^1*-norm. Then,* $\widehat{g}(\xi) = 2\pi i \xi_k \widehat{f}(\xi)$.

Proof. Properties (1.2) and (1.4) in Theorem 1.1 allow us to write

$$\left|\widehat{g}(\xi) - \widehat{f}(\xi)\frac{(1-e^{-2\pi i h(\xi\cdot e_k)})}{h}\right|,$$

then take $h \to 0$ to obtain the result. □

From the previous theorems it is easy to obtain the formulae:

$$\begin{aligned} P(D)\widehat{f}(\xi) &= (P(-2\pi i x)f(x))^{\wedge}(\xi), \\ (\widehat{P(D)f})(\xi) &= P(2\pi i\xi)\widehat{f}(\xi), \end{aligned} \tag{1.10}$$

where P is a polynomial in n variables and $P(D)$ denotes the differential operator associated to P.

Now we turn our attention to the following question: Given the Fourier transform $\widehat{f}$ of a function in $L^1(\mathbb{R}^n)$, how can one recover f?

Examples 1.3–1.5 suggest the use of the formula

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i (x\cdot\xi)} d\xi.$$

Unfortunately, $\widehat{f}(\xi)$ may be nonintegrable (see Example 1.1). To avoid this problem, one needs to use the so called method of summability (Abel and Gauss) similar to those used in the study of Fourier series. Combining the ideas behind the Gauss summation method and the identities (1.4), (1.7), (1.8), we obtain the following equalities:

$$\begin{aligned} f(x) &= \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x) = \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} \tau_x \frac{e^{-|y|^2/4t}}{(4\pi t)^{n/2}} f(y)dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} (\widehat{e^{2\pi i(x\cdot\xi)} e^{-4\pi^2 t|\xi|^2}})(y) f(y) dy \\ &= \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi)d\xi, \end{aligned}$$

where the limit is taken in the L^1-norm.

Thus, if f and $\widehat{f}$ are both integrable, the Lebesgue dominated convergence theorem guarantees the point-wise equality. Also, if $f \in L^1(\mathbb{R}^n)$ is continuous at the point x_0, we get:

$$f(x_0) = \lim_{t\to 0} \frac{e^{-|\cdot|^2/4t}}{(4\pi t)^{n/2}} * f(x_0) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i(x_0\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi)d\xi.$$

Collecting this information, we get the following result.

Proposition 1.2. *Let $f \in L^1(\mathbb{R}^n)$. Then,*

$$f(x) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi)d\xi,$$

where the limit is taken in the L^1-norm. Moreover, if f is continuous at the point x_0, then the following point-wise equality holds:

$$f(x_0) = \lim_{t\to 0} \int_{\mathbb{R}^n} e^{2\pi i(x_0\cdot\xi)} e^{-4\pi^2 t|\xi|^2} \widehat{f}(\xi)d\xi.$$

Let $f,\ \widehat{f} \in L^1(\mathbb{R}^n)$. *Then,*

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi)}\widehat{f}(\xi)d\xi, \quad \textit{almost everywhere } x \in \mathbb{R}^n.$$

From this result and Theorem 1.1 we can conclude that

$$\wedge\ :\ L^1(\mathbb{R}^n) \longrightarrow C_\infty(\mathbb{R}^n)$$

is a linear, one-to-one (Exercise 1.6 (i)), bounded map. However, it is not surjective (Exercise 1.6 (iii)).

1.2 The Fourier Transform in $L^2(\mathbb{R}^n)$

To define the Fourier transform in $L^2(\mathbb{R}^n)$, we shall first consider that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a dense subset of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Theorem 1.3 (Plancherel). *Let* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then,* $\widehat{f} \in L^2(\mathbb{R}^n)$ *and*

$$\|\widehat{f}\|_2 = \|f\|_2. \tag{1.11}$$

Proof. Let $g(x) = -x$. Using Young's inequality (1.39), (1.6), and Exercise 1.7 (ii), it follows that

$$f * g \in L^1(\mathbb{R}^n) \cap C_\infty(\mathbb{R}^n) \quad \text{and} \quad \widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

Since $\widehat{g} = \overline{(\widehat{f})}$, we find that $\widehat{(f * g)} = |\widehat{f}|^2 \geq 0$. Hence, $\widehat{(f * g)} \in L^1(\mathbb{R}^n)$ (see Exercise 1.7 (iii)). Proposition 1.2 shows that

$$(f * g)(0) = \int_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi,$$

and

$$\begin{aligned}\|\widehat{f}\|_2^2 &= \int_{\mathbb{R}^n} \widehat{(f * g)}(\xi)\, d\xi = (f * g)(0) \\ &= \int_{\mathbb{R}^n} f(x)g(0 - x)\, dx = \int_{\mathbb{R}^n} f(x)\bar{f}(x)\, dx = \|f\|_2^2.\end{aligned}$$

□

This result shows that the Fourier transform defines a linear bounded operator from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Indeed, this operator is an isometry. Thus, there is a unique bounded extension $\mathcal{F}$ defined in all $L^2(\mathbb{R}^n)$. $\mathcal{F}$ is called the Fourier transform in $L^2(\mathbb{R}^n)$. We shall use the notation $\widehat{f} = \mathcal{F}(f)$ for $f \in L^2(\mathbb{R}^n)$. In general, the definition $\widehat{f}$ is realized as a limit in L^2 of the sequence $\{\widehat{h}_j\}$, where $\{h_j\}$ denotes any sequence in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ that converges to f in the L^2-norm. It is convenient to take h_j equals f for $|x| \leq j$ and to have h_j vanishing for $|x| > j$. Then,

$$\widehat{h}_j(\xi) = \int_{|x|<j} f(x) e^{-2\pi i(x\cdot\xi)}\, dx = \int_{\mathbb{R}^n} h_j(x) e^{-2\pi i(x\cdot\xi)}\, dx$$

and so,

$$\widehat{h}_j(\xi) \to \widehat{f}(\xi) \quad \text{in } L^2, \quad \text{as } j \to \infty.$$

Example 1.6 Let $n = 1$ and $f(x) = \dfrac{1}{\pi}\dfrac{x}{1+x^2}$. Observe that $f \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. Differentiating the identity in the Example 1.5 with respect to a and taking $b = 1$ we get:

$$\int_{-\infty}^{\infty} \frac{x \sin(ax)}{1+x^2}\, dx = \pi e^{-a}, \qquad a > 0,$$

which combined with the previous remark gives:

$$\widehat{f}(\xi) = -i\, \mathrm{sgn}(\xi) e^{-2\pi|\xi|}.$$

A surjective isometry defines a "unitary operator." Theorem 1.3 affirms that $\mathcal{F}$ is an isometry. Let us see that $\mathcal{F}$ is also surjective.

Theorem 1.4. *The Fourier transform defines a unitary operator in* $L^2(\mathbb{R}^n)$.

Proof. From the identity (1.11) it follows that $\mathcal{F}$ is an isometry. In particular, its image is a closed subspace of $L^2(\mathbb{R}^n)$. Assume that this is a proper subspace of L^2. Then, there exists $g \neq 0$ such that

$$\int_{\mathbb{R}^n} \widehat{f}(y) g(y) dy = 0, \quad \text{for any } f \in L^2(\mathbb{R}^n).$$

Using formula (1.7; Theorem 1.7), which obviously extends to $f, g \in L^2(\mathbb{R}^n)$, we have that

$$\int_{\mathbb{R}^n} f(y)\widehat{g}(y) dy = \int_{\mathbb{R}^n} \widehat{f}(y) g(y) dy = 0, \quad \text{for any } f \in L^2.$$

Therefore, $\widehat{g}(\xi) = 0$ almost everywhere, which contradicts

$$\|g\|_2 = \|\widehat{g}\|_2 \neq 0.$$

□

Theorem 1.5. *The inverse of the Fourier transform* $\mathcal{F}^{-1}$ *can be defined by the formula*

$$\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x), \quad \textit{for any} \quad f \in L^2(\mathbb{R}^n). \tag{1.12}$$

Proof. $\mathcal{F}^{-1}\widehat{f} = \tilde{f}$ is the limit in the L^2-norm of the sequence

$$f_j(x) = \int\limits_{|\xi|<j} \widehat{f}(\xi)e^{2\pi i(\xi \cdot x)}d\xi.$$

First, we consider the case where $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. It suffices to verify that this agrees with $\mathcal{F}^*\widehat{f}$, where $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$ (we recall the fact that for a unitary operator the adjoint and the inverse are equal). This can be checked as follows:

$$\tilde{f}(x) = \int\limits_{\mathbb{R}^n} \widehat{f}(\xi)e^{2\pi i(\xi \cdot x)}d\xi = \lim_{j\to\infty} f_j(x) \quad \text{in} \quad L^2(\mathbb{R}^n),$$

and

$$(g, \tilde{f}) = \int\limits_{\mathbb{R}^n} g(x)\overline{\Big(\int\limits_{\mathbb{R}^n} \widehat{f}(\xi)\, e^{2\pi i(\xi \cdot x)}\, d\xi\Big)}dx$$

$$= \int\limits_{\mathbb{R}^n}\Big(\int\limits_{\mathbb{R}^n} g(x)e^{-2\pi i(x \cdot \xi)}dx\Big)\overline{\widehat{f}(\xi)}d\xi = (\mathcal{F}g, \widehat{f})$$

for any $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Hence $\tilde{f} = f$.

The general case follows by combining the above result and an argument involving a justification of passing to the limit. □

1.3 Tempered Distributions

From the definitions of the Fourier transform on $L^1(\mathbb{R}^n)$ and on $L^2(\mathbb{R}^n)$, there is a natural extension to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. It is not hard to see that $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ contains the spaces $L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2$. On the other hand, as we shall prove, any function in $L^p(\mathbb{R}^n)$ for $p > 2$ has a Fourier transform in the distribution sense. However, they may not be function, they are *tempered distributions*. Before studying them, it is convenient to see how far Definition 1.1 can be carried out.

Example 1.7 Let $n \geq 1$ and $f(x) = \delta_0$, the delta function, i.e., the measure of mass one concentrated at the origin. Using (1.1) one finds that

$$\widehat{\delta_0}(\xi) = \int\limits_{\mathbb{R}^n} \delta_0(x)\, e^{-2\pi i(x \cdot \xi)}dx \equiv 1.$$

In fact, Definition 1.1 tells us that if μ is a bounded measure, then $\widehat{\mu}(\xi)$ represents a function in $L^\infty(\mathbb{R}^n)$.

Suppose that given $f(x) \equiv 1$ we want to find $\widehat{f}(\xi)$. In this case, Definition 1.1 cannot be used directly. It is necessary to introduce the notion of tempered distribution. For this purpose, we first need the following family of seminorms.

For each $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ we denote the seminorm $|\!|\!| \cdot |\!|\!|_{(\nu,\beta)}$ defined as:

$$|\!|\!| f |\!|\!|_{(\nu,\beta)} = \|x^\nu \partial_x^\beta f\|_\infty .$$

Now we can define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of the C^∞-functions decaying at infinity, i.e.,

$$\mathcal{S}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : |\!|\!| \varphi |\!|\!|_{(\nu,\beta)} < \infty \quad \text{for any} \quad \nu, \beta \in (\mathbb{Z}^+)^n\}.$$

Thus, $C_0^\infty(\mathbb{R}^n) \subsetneq \mathcal{S}(\mathbb{R}^n)$ (consider $f(x)$ as in Example 1.3).

The topology in $\mathcal{S}(\mathbb{R}^n)$ is given by the family of seminorms $|\!|\!| \cdot |\!|\!|_{(\nu,\beta)}$, $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$.

Definition 1.3. Let $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n)$. Then, $\varphi_j \to 0$ as $j \to \infty$, if for any $(\nu, \beta) \in (\mathbb{Z}^+)^{2n}$ one has that

$$|\!|\!| \varphi_j |\!|\!|_{(\nu,\beta)} \longrightarrow 0 \quad \text{as} \quad j \to \infty.$$

The relationship between the Fourier transform and the function space $\mathcal{S}(\mathbb{R}^n)$ is described in the formulae (1.10). More precisely, we have the following result (see Exercise 1.13).

Theorem 1.6. *The map* $\varphi \mapsto \widehat{\varphi}$ *is an isomorphism from* $\mathcal{S}(\mathbb{R}^n)$ *into itself.*

Thus, $\mathcal{S}(\mathbb{R}^n)$ appears naturally associated to the Fourier transform. By duality, we can define the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.4. We say that $\Psi : \mathcal{S}(\mathbb{R}^n) \mapsto \mathbb{C}$ defines a tempered distribution, i.e., $\Psi \in \mathcal{S}'(\mathbb{R}^n)$ if:

1. Ψ is linear.
2. Ψ is continuous, i.e., if for any $\{\varphi_j\} \subseteq \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_j \to 0$ as $j \to \infty$, then the numerical sequence $\Psi(\varphi_j) \to 0$ as $j \to \infty$.

It is easy to check that any bounded function f defines a tempered distribution Ψ_f, where

$$\Psi_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \text{for any} \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.13}$$

In fact, this identity allows us to see that any locally integrable function with polynomial growth at infinity defines a tempered distribution. In particular, we have the $L^p(\mathbb{R}^n)$ spaces with $1 \le p \le \infty$. The following example gives us a tempered distribution outside these function spaces.

Example 1.8 In $\mathcal{S}'(\mathbb{R})$, define the *principal value function* of $1/x$, denoted by p.v. $\frac{1}{x}$, by the expression

$$\text{p.v.}\,\frac{1}{x}(\varphi) = \lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\varphi(x)}{x}dx,$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$. Since $1/x$ is an odd function,

$$\text{p.v.}\frac{1}{x}(\varphi) = \int\limits_{|x|<1} \frac{\varphi(x)-\varphi(0)}{x}dx + \int\limits_{|x|>1} \frac{\varphi(x)}{x}dx. \tag{1.14}$$

Therefore,

$$\left|\text{p.v.}\,\frac{1}{x}(\varphi)\right| \le 2\|\varphi'\|_\infty + 2\|x\varphi\|_\infty, \tag{1.15}$$

and consequently, p.v. $\frac{1}{x} \in \mathcal{S}'(\mathbb{R})$.

Now, given a $\Psi \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform can be defined in the following natural form.

Definition 1.5. Given $\Psi \in \mathcal{S}'(\mathbb{R}^n)$, its Fourier transform $\widehat{\Psi} \in \mathcal{S}'(\mathbb{R}^n)$ is defined as:

$$\widehat{\Psi}(\varphi) = \Psi(\widehat{\varphi}), \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.16}$$

Observe that for $f \in L^1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, (1.7), (1.13), and (1.16) tell us that

$$\widehat{\Psi}_f(\varphi) = \Psi_f(\widehat{\varphi}) = \int\limits_{\mathbb{R}^n} f(x)\widehat{\varphi}(x)dx = \int\limits_{\mathbb{R}^n} \widehat{f}(x)\varphi(x)dx = \Psi_{\widehat{f}}(\varphi).$$

Therefore, for $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ one has that $\widehat{\Psi}_f = \Psi_{\widehat{f}}$. Thus, Definition 1.5 is consistent with the theory of the Fourier transform developed in Sects. 1.1 and 1.2.

Example 1.9 Let $f(x) \equiv 1 \in L^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. Using the previous notation, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that

$$\widehat{\Psi}_1(\varphi) = \Psi_1(\widehat{\varphi}) = \int\limits_{\mathbb{R}^n} 1\, \widehat{\varphi}(x)dx = \varphi(0) = \int\limits_{\mathbb{R}^n} \delta_0(x)\,\varphi(x)dx = \delta_0(\varphi).$$

Hence $\widehat{1} = \delta_0$. We recall that in Example 1.7 we already saw that $\widehat{\delta_0} = 1$.

Next we compute the Fourier transform of the tempered distribution in Example 1.8.

Example 1.10 Combining Definition 1.5, Fubini's theorem, and the Lebesgue dominated convergence theorem we have that for any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}\widehat{\text{p.v.}\,\frac{1}{x}}(\varphi) &= \text{p.v.}\,\frac{1}{x}(\widehat{\varphi}) = \lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{\widehat{\varphi}(x)}{x}dx \\ &= \lim_{\epsilon\downarrow 0} \int\limits_{\epsilon<|x|<1/\epsilon} \frac{1}{x}\Big(\int_{-\infty}^{\infty} \varphi(y)e^{-2\pi i x y}\,dy\Big)dx\end{aligned}$$

$$= \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} \varphi(y)\Big(\int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx\Big)\, dy$$

$$= \int_{-\infty}^{\infty} \varphi(y)\Big(\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx\Big) dy$$

$$= -i\pi \int_{-\infty}^{\infty} \operatorname{sgn}(y)\, \varphi(y) dy,$$

where a change of variables and complex integration have been used to conclude that

$$\lim_{\epsilon\downarrow 0} \int_{\epsilon<|x|<1/\epsilon} \frac{e^{-2\pi ixy}}{x} dx = -2i \int_0^{\infty} \frac{\sin(2\pi xy)}{x} dx = -2i \operatorname{sgn}(y) \int_0^{\infty} \frac{\sin(x)}{x} dx$$

$$= -i\pi \operatorname{sgn}(y).$$

This yields the identity:

$$\widehat{\text{p.v.} \frac{1}{x}}(\xi) = -i\pi \operatorname{sgn}(\xi).$$

The topology in $\mathcal{S}'(\mathbb{R}^n)$ can be described in the following form.

Definition 1.6. Let $\{\Psi_j\} \subset \mathcal{S}'(\mathbb{R}^n)$. Then, $\Psi_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$, if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ it follows that $\Psi_j(\varphi) \longrightarrow 0$ as $j \to \infty$.

As a consequence of the Definitions 1.4, 1.6, we get the next extension of Theorem 1.6, whose proof we leave as an exercise.

Theorem 1.7. *The map $\mathcal{F} : \Psi \mapsto \widehat{\Psi}$ is an isomorphism from $\mathcal{S}'(\mathbb{R}^n)$ into itself.*

Combining the above results with an extension of Example 1.3 (see Exercise 1.2), we can justify the following computation related with the fundamental solution of the time-dependent Schrödinger equation.

Example 1.11 $\widehat{e^{-4\pi^2 it|x|^2}} = \lim_{\epsilon\to 0^+} \widehat{e^{-4\pi^2(\epsilon+it)|x|^2}}$ in $\mathcal{S}'(\mathbb{R}^n)$.

From Exercise 1.2, it follows that

$$(\widehat{e^{-4\pi^2(\epsilon+it)|x|^2}})(\xi) = \frac{e^{-|\xi|^2/4(\epsilon+it)}}{[4\pi(\epsilon+it)]^{n/2}}.$$

Taking the limit $\epsilon \to 0^+$, we obtain:

$$(\widehat{e^{-4\pi^2 it|x|^2}})(\xi) = \frac{e^{i|\xi|^2/4t}}{(4\pi it)^{n/2}}. \tag{1.17}$$

As an application of these ideas, we introduce the Hilbert transform.

Definition 1.7. For $\varphi \in \mathcal{S}(\mathbb{R})$, we define its *Hilbert transform* $\mathsf{H}(\varphi)$ by

$$\mathsf{H}(\varphi)(y) = \frac{1}{\pi}\,\text{p.v.}\frac{1}{x}\,(\varphi(y-\cdot)) = \frac{1}{\pi}\,\text{p.v.}\,\frac{1}{x} * \varphi(y).$$

From (1.14) and (1.15) it is clear that $\mathsf{H}(\varphi)(y)$ is defined for any $y \in \mathbb{R}$ and it is bounded by $g(y) = a|y| + b$, with $a, b > 0$ depending on φ. In particular, we have that $\mathsf{H}(\varphi) \in \mathcal{S}'(\mathbb{R})$. Let us compute its Fourier transform.

Example 1.12 From Example 1.10 and the identity

$$\mathsf{H}(\varphi)(y) = \lim_{\epsilon \to 0}\Big(\frac{1}{\pi}\frac{1}{x}\chi_{\{\epsilon<|x|<1/\epsilon\}} * \varphi\Big)(y) \quad \text{in} \quad \mathcal{S}'(\mathbb{R})$$

it follows that

$$\lim_{\epsilon \to 0}\Big(\frac{1}{\pi}\,\frac{1}{x}\,\widehat{\chi_{\{\epsilon<|x|<1/\epsilon\}}} * \varphi\Big)(\xi) = -i\,\text{sgn}(\xi)\,\widehat{\varphi}(\xi).$$

This implies that

$$\widehat{\mathsf{H}(\varphi)}(\xi) = -i\,\text{sgn}(\xi)\,\widehat{\varphi}(\xi), \quad \text{for any } \varphi \in \mathcal{S}(\mathbb{R}). \tag{1.18}$$

The identity (1.18) allows us to extend the Hilbert transform as an isometry in $L^2(\mathbb{R})$. It is not hard to see that

$$\|\mathsf{H}(\varphi)\|_2 = \|\varphi\|_2 \quad \text{and} \quad \mathsf{H}(\mathsf{H}(\varphi)) = -\varphi.$$

Other properties of the Hilbert transform are deduced in the exercises in Chaps. 1 and 2.

In Definition 1.7, we have implicitly utilized the following result, which is employed again in the applications at the end of this chapter.

Proposition 1.3. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\Psi \in \mathcal{S}'(\mathbb{R}^n)$. *Define*

$$\Psi * \varphi(x) = \Psi(\varphi(x-\cdot)). \tag{1.19}$$

Then,

$$\Psi * \varphi \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$$

and

$$\widehat{\Psi * \varphi} = \widehat{\Psi}\widehat{\varphi}, \tag{1.20}$$

where $\widehat{\Psi}\widehat{\varphi} \in \mathcal{S}'(\mathbb{R}^n)$ *is defined as* $\widehat{\Psi}\widehat{\varphi}(\phi) = \widehat{\Psi}(\widehat{\varphi}\phi)$ *for any* $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. It is left as an exercise. □

1.4 Oscillatory Integrals in One Dimension

In many problems and applications the following question arises:

What is the asymptotic behavior of $I(\lambda)$ when $\lambda \to \infty$, where

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx, \tag{1.21}$$

and ϕ is a smooth real-valued function, called the "phase function," and f is a smooth complex-valued function?

We shall see that this asymptotic behavior is determined by the points $\bar{x}$, where the derivative of ϕ vanishes, i.e., $\phi'(\bar{x}) = 0$.

Proposition 1.4. *Let $f \in C_0^\infty([a,b])$ and $\phi'(x) \neq 0$ for any $x \in [a,b]$. Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x)dx = O(\lambda^{-k}), \quad as\ \lambda \to \infty \tag{1.22}$$

for any $k \in \mathbb{Z}^+$.

Proof. Define the differential operator

$$\mathcal{L}(f) = \frac{1}{i\lambda\phi'}\frac{df}{dx},$$

which satisfies

$$\mathcal{L}^t(f) = -\frac{d}{dx}\left(\frac{f}{i\lambda\phi'}\right) \quad \text{and} \quad \mathcal{L}^k(e^{i\lambda\phi}) = e^{i\lambda\phi},$$

where $\mathcal{L}^t$ denotes the adjoint of $\mathcal{L}$. Using integration by parts it follows that

$$\int_a^b e^{i\lambda\phi} f\,dx = \int_a^b \mathcal{L}^k(e^{i\lambda\phi}) f\,dx$$

$$= (-1)^k \int_a^b e^{i\lambda\phi}(\mathcal{L}^t)^k f\,dx = O(\lambda^{-k}), \quad \text{as } \lambda \to \infty.$$

□

Proposition 1.5. *Let $k \in \mathbb{Z}^+$ and $|\phi^{(k)}(x)| \geq 1$ for any $x \in [a,b]$ with $\phi'(x)$ monotonic in the case $k = 1$. Then,*

$$\left|\int_a^b e^{i\lambda\phi(x)} dx\right| \leq c_k\lambda^{-1/k}, \tag{1.23}$$

where the constant c_k *is independent of* a, b.

Proof. For $k = 1$, we have that

$$\int_a^b e^{i\lambda\phi} dx = \int_a^b \mathcal{L}(e^{i\lambda\phi}) dx = \frac{1}{i\lambda\phi'} e^{i\lambda\phi}\Big|_a^b - \int_a^b e^{i\lambda\phi} \frac{1}{i\lambda} \frac{d}{dx}\Big(\frac{1}{\phi'}\Big) dx.$$

Clearly, the first term on the right-hand side is bounded by $2\lambda^{-1}$. On the other hand, the hypothesis of monotonicity on ϕ' guarantees that

$$\Big|\int_a^b e^{i\lambda\phi} \frac{1}{i\lambda} \frac{d}{dx}\Big(\frac{1}{\phi'}\Big) dx\Big| \le \frac{1}{\lambda} \int_a^b \Big|\frac{d}{dx}\Big(\frac{1}{\phi'}\Big)\Big| dx$$
$$= \frac{1}{\lambda}\Big|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\Big| \le \frac{2}{\lambda}.$$

This yields the proof of the case $k = 1$.

For the proof of the case $k \ge 2$, induction in k is used. Assuming the result for k, we shall prove it for $k + 1$. By hypothesis, $|\phi^{(k+1)}(x)| \ge 1$. Let $x_0 \in [a, b]$ be such that

$$|\phi^{(k)}(x_0)| = \min_{a \le x \le b} |\phi^{(k)}(x)|.$$

If $\phi^{(k)}(x_0) = 0$, outside the interval $(x_0 - \delta, x_0 + \delta)$, one has that $|\phi^{(k)}(x)| \ge \delta$, with ϕ' monotonic if $k = 1$. Splitting the domain of integration and applying the hypothesis we obtain that

$$\Big|\int_a^{x_0-\delta} e^{i\lambda\phi(x)} dx\Big| + \Big|\int_{x_0+\delta}^b e^{i\lambda\phi(x)} dx\Big| \le c_k(\lambda\delta)^{-1/k}.$$

A simple computation shows that

$$\Big|\int_{x_0-\delta}^{x_0+\delta} e^{i\lambda\phi(x)} dx\Big| \le 2\delta.$$

Thus,

$$\Big|\int_a^b e^{i\lambda\phi(x)} dx\Big| \le c_k(\lambda\delta)^{-1/k} + 2\delta.$$

If $\phi^{(k)}(x_0) \ne 0$, then $x_0 = a$ or b and a similar argument provides the same bound. Finally, taking $\delta = \lambda^{-1/(k+1)}$ we complete the proof. □

Corollary 1.1 (van der Corput). *Under the hypotheses of Proposition 1.5,*

$$\Big|\int_a^b e^{i\lambda\phi(x)} f(x)\, dx\Big| \le c_k\lambda^{-1/k} \left(\|f\|_\infty + \|f'\|_1\right) \qquad (1.24)$$

with c_k *independent of* a, b.

Proof. Define

$$G(x) = \int_a^x e^{i\lambda\phi(y)}\,dy.$$

By (1.23) one has that

$$|G(x)| \leq c_k \lambda^{-1/k}.$$

Now using integration by parts we obtain:

$$\left|\int_a^b e^{i\lambda\phi} f\,dx\right| = \left|\int_a^b G' f\,dx\right| \leq \left|(Gf)\Big|_a^b\right| + \left|\int_a^b G f'\,dx\right|$$
$$\leq c_k \lambda^{-1/k}\left(\|f\|_\infty + \|f'\|_1\right).$$

□

Next, we shall study an application of these results.

Proposition 1.6. *Let* $\beta \in [0, 1/2]$ *and* $I_\beta(x)$ *be the oscillatory integral*

$$I_\beta(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\,|\eta|^\beta d\eta. \tag{1.25}$$

Then, $I_\beta \in L^\infty(\mathbb{R})$.

Proof. First, we fix $\varphi_0 \in C^\infty(\mathbb{R})$ such that

$$\varphi_0(\eta) = \begin{cases} 1, & \text{if } |\eta| > 2 \\ 0, & \text{if } |\eta| < 1. \end{cases}$$

Observe that $(1-\varphi_0)(\eta)e^{i\eta^3}|\eta|^\beta \in L^1(\mathbb{R})$, therefore its Fourier transform belongs to $L^\infty(\mathbb{R})$. Thus, it suffices to consider

$$\tilde{I}_\beta(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\,|\eta|^\beta \varphi_0(\eta) d\eta.$$

For $x \geq -3$, the phase function $\phi_x(\eta) = x\eta + \eta^3$, in the support of φ_0, satisfies

$$|\phi_x'(\eta)| = |x + 3\eta^2| \geq (|x| + |\eta|^2).$$

In this case, integration by parts leads to the desired result.

For $x < -3$, we consider the functions $(\varphi_1, \varphi_2) \in C_0^\infty \times C^\infty$ such that $\varphi_1(\eta) + \varphi_2(\eta) = 1$ with

$$\operatorname{supp}\varphi_1 \subset A = \left\{\eta : |x + 3\eta^2| \leq \frac{|x|}{2}\right\},$$

and

$$\varphi_2 = 0 \quad \text{in} \quad B = \left\{\eta : |x + 3\eta^2| < \frac{|x|}{3}\right\},$$

and we split the integral $\tilde{I}_\beta(x)$ in two pieces,

$$|\tilde{I}_\beta(x)| \leq |\tilde{I}^1_\beta(x)| + |\tilde{I}^2_\beta(x)|,$$

where

$$\tilde{I}^j_\beta(x) = \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta \varphi_0(\eta)\varphi_j(\eta) d\eta, \quad \text{for} \quad j = 1, 2.$$

When $\varphi_2(\eta) \neq 0$, the triangle inequality shows that

$$|\phi'_x(\eta)| = |x + 3\eta^2| \geq \frac{3}{13}(|x| + |\eta|^2).$$

Integration by parts leads to

$$|\tilde{I}^2_\beta(x)| = \left| \int_{-\infty}^{\infty} \frac{|\eta|^\beta}{\phi'_x(\eta)} \varphi_0(\eta)\varphi_2(\eta) \frac{d}{d\eta} e^{i(x\eta+\eta^3)}\, d\eta \right| \leq 100.$$

Now, if $\eta \in A$, we have that

$$\frac{|x|}{2} \leq 3\eta^2 \leq 3\frac{|x|}{2} \quad \text{and} \quad \left|\frac{d^2\phi_x}{d\eta^2}(\eta)\right| = 6|\eta| \geq |x|^{1/2}.$$

Thus (1.24) (van der Corput) and the form of φ_0, φ_1 guarantee the existence of a constant c independent of $x < -3$ such that

$$|\tilde{I}^1_\beta(x)| = \left| \int_{-\infty}^{\infty} e^{i(x\eta+\eta^3)}\, |\eta|^\beta \varphi_0(\eta)\varphi_1(\eta)\, d\eta \right| \leq c\, |x|^{-1/4} |x|^{\beta/2}.$$

□

1.5 Applications

Consider the initial value problem (IVP) for the linear Schrödinger equation:

$$\begin{cases} \partial_t u = i\Delta u, \\ u(x,0) = u_0(x), \end{cases} \tag{1.26}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Taking the Fourier transform with respect to the space variable x in (1.26) we obtain:

$$\begin{cases} \widehat{\partial_t u}(\xi,t) = \partial_t \widehat{u}(\xi,t) = \widehat{i\Delta u}(\xi,t) = -4\pi^2 i|\xi|^2\widehat{u}(\xi,t) \\ \widehat{u}(\xi,0) = \widehat{u}_0(\xi). \end{cases}$$

The solution of this family of ordinary differential equations (ODE), with parameter ξ, can be written as:

$$\widehat{u}(\xi,t) = e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi).$$

By Proposition 1.3 it follows that

$$\begin{aligned} u(x,t) &= (e^{-4\pi^2 it|\xi|^2}\widehat{u}_0(\xi))^{\vee} = (e^{-4\pi^2 it|\xi|^2})^{\vee} * u_0(x) \\ &= \frac{e^{i|\cdot|^2/4t}}{(4\pi it)^{n/2}} * u_0(x) = e^{it\Delta}u_0(x), \end{aligned} \tag{1.27}$$

where we have introduced the notation $e^{it\Delta}$ which is justified in Chapter 4.

Next, we consider the IVP associated to the linearized Korteweg–de Vries (KdV) equation:

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \\ v(x,0) = v_0(x) \end{cases} \tag{1.28}$$

for $t, x \in \mathbb{R}$. The previous argument shows that

$$v(x,t) = S_t * v_0(x) = (e^{8\pi^3 it\xi^3}\widehat{v}_0)^{\vee} = V(t)v_0(x), \tag{1.29}$$

where the kernel $S_t(x)$ is defined by the oscillatory integral:

$$S_t(x) = \int_{-\infty}^{\infty} e^{2\pi ix\xi}e^{8\pi^3 it\xi^3}\,d\xi. \tag{1.30}$$

After changing variables,

$$S_t(x) = \frac{1}{\sqrt[3]{3t}}\,Ai\left(\frac{x}{\sqrt[3]{3t}}\right), \tag{1.31}$$

where $Ai(\cdot)$ denotes the Airy function:

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\xi x+\xi^3/3)}d\xi. \tag{1.32}$$

By combining Proposition 1.6 (with $\beta = 0$) and a new change of variable we find that

$$\|S_t\|_\infty \le c|t|^{-1/3}. \tag{1.33}$$

Moreover, if $\beta \in [0, 1/2]$, then

$$\|D_x^\beta S_t\|_\infty \le c|t|^{-(\beta+1)/3}. \tag{1.34}$$

Hence, using Exercise 1.6 it follows that

$$\|D_x^\beta V(t)v_0\|_\infty = \|D_x^\beta S_t * v_0\|_\infty \le c|t|^{-(\beta+1)/3}\|v_0\|_1, \tag{1.35}$$

where $D_x^\beta = D^\beta = (-\Delta)^{\beta/2}$ denotes the homogeneous fractional derivative of order β, i.e.,

$$D^\beta f(x) = [(2\pi|\xi|)^\beta \widehat{f}(\xi)]^\vee(x). \tag{1.36}$$

Notice that the derivative of the phase function in (1.32) $\phi(\xi) = \xi x + \xi^3/3$ does not vanish for $x > 0$, i.e., $|\phi'(\xi)| = |x + \xi^2| \ge |x|$, so using Proposition 1.4 one sees that $Ai(x)$ has fast decay for $x > 0$. In fact, one has (see [Ho2] or [SSS]) that

$$|Ai(x)| \le \frac{1}{(1+x_-)^{1/4}}\, e^{-cx_+^{3/2}}, \tag{1.37}$$

and

$$|Ai'(x)| \le (1+x_-)^{1/4} e^{-cx_+^{3/2}}, \tag{1.38}$$

where $x_+ = \max\{x; 0\}$ and $x_- = \max\{-x; 0\}$.

Hence, (1.34) with $\beta = 1/2$ can be seen as an interpolation between (1.37) and (1.38) and the scaling.

Remark 1.1. The relevant references used in this chapter are the books [SW], [S2], [S3], [Sa], [Du], and [Rd].

1.6 Exercises

1.1 (i) Let $n \ge 1$ and $f(x) = e^{-2\pi|x|}$. Show that

$$\hat{f}(\xi) = \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}} \frac{1}{(1+|\xi|^2)^{(n+1)/2}}.$$

Hint: From the formula of Example 1.5 with $a = \beta$ and $b = 1$ one sees that

$$e^{-\beta} = \frac{2}{\pi}\int_0^\infty \frac{\cos(\beta x)}{1+x^2}\,dx,$$

which, combined with the equality:

$$\frac{1}{1+x^2} = \int_0^\infty e^{-(1+x^2)\rho}\,d\rho, \quad \text{yields} \quad e^{-\beta} = \int_0^\infty \frac{e^{-\rho}}{\sqrt{\rho}}\, e^{-\beta^2/4\rho}\,d\rho.$$

Use this identity to obtain the desired result.

(ii) Let $n = 1$ and $f(x) = \dfrac{1}{\pi}\dfrac{1}{(1+x^2)^2}$. Show that

$$\hat{f}(\xi) = \frac{1}{2}e^{-2\pi|\xi|}(2\pi|\xi|+1).$$

Hint: Differentiate the identity in Example 1.5.

1.2 (i) Prove the following extension in $\mathcal{S}'(\mathbb{R}^n)$ of formula (1.8):

$$\widehat{(e^{-a|x|^2})}(\xi) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|\xi|^2/a}, \quad \mathcal{R}e\, a \geq 0, \ a \neq 0,$$

where $\sqrt{a}$ is defined as the branch with $\mathcal{R}e\, a > 0$.
Hint: Use an analytic continuation argument.

(ii) Show that if $a = 1 + it$, then

$$\left\| \left(\frac{\pi}{a}\right)^{n/2} e^{-\pi^2|x|^2/a} \right\|_p \sim c_p (1+t)^{n(\frac{1}{p}-\frac{1}{2})}, \quad 1 \leq p \leq \infty, \ t > 0,$$

and

$$\| e^{-\pi a|\xi|^2} \|_q \sim c_q, \quad 1 \leq q \leq \infty,$$

where $f(t) \sim g(t)$, for $f, g \geq 0$, means that there exists $c > 1$ such that

$$c^{-1} f(t) \leq g(t) \leq c\, f(t), \quad \forall t > 0.$$

1.3 Prove Young's inequality: Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and $g \in L^1(\mathbb{R}^n)$. Then, $f * g \in L^p(\mathbb{R}^n)$ with

$$\| f * g \|_p \leq \| f \|_p \| g \|_1. \tag{1.39}$$

1.4 Prove the Minkowski integral inequality. If $1 \leq p \leq \infty$, then

$$\left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x,y)\, dx \right|^p dy \right)^{1/p} \leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| f(x,y) \right|^p dy \right)^{1/p} dx. \tag{1.40}$$

Observe that the proof of the cases $p = 1, \infty$ is immediate.

1.5 Let $f \in L^p((0,\infty))$, $1 < p < \infty$, $f \geq 0$:

(i) Prove Hardy's inequality:

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(s)\, ds \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty (f(x))^p\, dx. \tag{1.41}$$

(ii) Prove that equality in (1.41) holds if and only if $f \equiv 0$, a.e., and that the constant $c_p = p/(p-1)$ is optimal in (1.41).

(iii) Prove that (1.41) fails for $p = 1$ and $p = \infty$.
Hint: Assuming $f \in C_0((0,\infty))$ define

$$F(x) = \frac{1}{x}\int_0^\infty f(s)\,ds, \text{so } x\,F' = f - F.$$

Use integration by parts and the Hölder inequality to obtain (1.41).

1.6 Consider the Fourier transform $\widehat{\ }$ as a map from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$.

(i) Prove that $\widehat{\ }$ is injective.

(ii) Prove that the image of $\widehat{\ }$, i.e., $\widehat{L^1(\mathbb{R}^n)}$, is an algebra with respect to the point-wise multiplication of functions.

(iii) Prove that $\widehat{L^1(\mathbb{R}^n)} \subsetneq C_\infty(\mathbb{R}^n)$, where $C_\infty(\mathbb{R}^n)$ denotes the space of continuous functions vanishing at infinity.

Hint: From Example 1.2 we have that $\|g_k\|_\infty = 2$ and

$$\lim_{k\uparrow\infty} \|\widehat{g_k}\|_1 = \infty.$$

Apply the open mapping theorem to get the desired result.

1.7 (i) Prove the following generalization of (1.6) in Theorem 1.1:

If $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then $\widehat{(f*g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.

(ii) If $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$, with $1/p + 1/p' = 1$, $1 < p < \infty$, then $f * g \in C_\infty(\mathbb{R}^n)$. What can you affirm if $p = 1, \infty$?

(iii) If $f \in L^1(\mathbb{R}^n)$, with f continuous at the point 0 and $\widehat{f} \ge 0$, then $\widehat{f} \in L^1(\mathbb{R}^n)$.

Hint: Use Proposition 1.2 and Fatou's lemma.

1.8 Show that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2} \quad\text{and}\quad \int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}.$$

Hint: Combine the identities (1.7), (1.11), and Example 1.1.

1.9 For a given $f \in L^2(\mathbb{R}^n)$ prove that the following statements are equivalent:

(i) $g \in L^2(\mathbb{R}^n)$ is the partial derivative of $f \in L^2(\mathbb{R}^n)$ with respect to the kth variable according to Definition 1.2.

(ii) There exists $g \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)\partial_{x_k}\phi(x)\,dx = -\int_{\mathbb{R}^n} g(x)\phi(x)\,dx \tag{1.42}$$

for any $\phi \in C_0^\infty(\mathbb{R}^n)$. In general, if (1.42) holds for two distributions f, g, then one says that g is the kth partial derivative of f in the distribution sense.

(iii) There exists $\{f_j\} \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\|f_j - f\|_2 \to 0 \quad \text{as} \quad j \to \infty,$$

and $\{\partial_{x_k} f_j\}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$.

(iv) $\xi_k \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$.

(v)

$$\sup_{h>0} \int_{\mathbb{R}^n} \left| \frac{f(x+he_k) - f(x)}{h} \right|^2 dx < \infty.$$

For $p \neq 2$, which of the above statements are still equivalent?

1.10 (Paley–Wiener theorem) Prove that if $f \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| \leq M\}$, then $\widehat{f}(\xi)$ can be extended analytically to $\mathbb{C}^n$. Moreover, if $k \in \mathbb{Z}^+$ one has that

$$|\widehat{f}(\xi + i\eta)| \leq c_k \frac{e^{2\pi M|\eta|}}{(1 + |(\xi + i\eta)|)^k} \quad \text{for any } \xi + i\eta \in \mathbb{C}^n. \tag{1.43}$$

Prove the converse, i.e., if $F(\xi + i\eta)$ is an analytic function in $\mathbb{C}^n$ satisfying (1.43), then F is the Fourier transform of some $f \in C_0^\infty(\mathbb{R}^n)$ with support in $\{x \in \mathbb{R}^n : |x| \leq M\}$.

1.11 Show that if $f \in L^1(\mathbb{R}^n)$, $f \not\equiv 0$, with compact support, then for any $\epsilon > 0$, $\hat{f} \notin L^1(e^{\epsilon|x|}dx)$.

1.12 Prove that given $k \in \mathbb{Z}^+$ and $a_\alpha \in \mathbb{R}^k$, with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$, there exists $f \in C_0^\infty(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} x^\alpha f(x)\,dx = a_\alpha.$$

Hint: Use Exercise 1.10.

1.13 (i) Prove that if $f, g \in \mathcal{S}$, then $f * g \in \mathcal{S}$.

(ii) Prove that the Fourier transform is an isomorphism from $\mathcal{S}$ into itself.

(iii) Using the results in Section 1.3, find explicitly $\Psi = \widehat{|x|^2} \in \mathcal{S}'(\mathbb{R}^n)$.

(iv) Prove Proposition 1.3.

1.14 In this problem we shall prove that

$$\widehat{\frac{1}{|x|^\alpha}}(\xi) = c_{n,\alpha} \frac{1}{|\xi|^{n-\alpha}} \qquad \text{for } \alpha \in (0, n)$$

as a tempered distribution, i.e., $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int \frac{1}{|x|^\alpha} \widehat{\varphi}(x)\,dx = c_{n,\alpha} \int \frac{1}{|\xi|^{n-\alpha}} \varphi(\xi)\,d\xi, \tag{1.44}$$

where $c_{n,\alpha} = \pi^{\alpha - n/2}\, \Gamma(n/2 - \alpha/2)/\Gamma(\alpha/2)$.

(i) Combining the Parseval identity and Example 1.3 show that for $\delta > 0$

$$\int e^{-\pi\delta|x|^2} \widehat{\varphi}(x)\,dx = \delta^{-n/2} \int e^{-\pi|x|^2/\delta} \varphi(x)\,dx. \tag{1.45}$$

(ii) Prove the formula

$$\int_0^\infty e^{-\pi\delta|x|^2} \delta^{\beta-1}\,d\delta = \frac{c_\beta}{|x|^{2\beta}} \quad \text{for any } \beta > 0. \tag{1.46}$$

(iii) Multiply both sides of (1.45) by $\delta^{\frac{n-\alpha}{2}-1}$, integrate on δ, use Fubini's theorem and (1.46) to get (1.44).

1.15 Prove the following identities, where H denotes the Hilbert transform:

(i) $\mathsf{H}(fg) = \mathsf{H}(f)g + f\mathsf{H}(g) + \mathsf{H}(\mathsf{H}(f)\mathsf{H}(g))$.

(ii) $\mathsf{H}(\chi_{(-1,1)})(x) = \dfrac{1}{\pi}\log\left|\dfrac{x+1}{x-1}\right|$.

(iii) $\mathsf{H}\left(\dfrac{a}{x^2+a^2}\right) = \dfrac{x}{x^2+a^2}, \quad a > 0.$

1.16 Prove that if $\varphi \in \mathcal{S}(\mathbb{R})$, then $\mathsf{H}(\varphi) \in L^1(\mathbb{R})$ if and only if $\widehat{\varphi}(0) = 0$.

1.17 Consider the function $f_a(x) = \dfrac{x}{a-x^2}$.

(i) If $a \geq 0$ prove that the principal value function of $f_a(x)$,

$$\text{p.v.}\frac{x}{a-x^2}(\varphi) = \lim_{\epsilon\downarrow 0}\int\limits_{\epsilon<|a-x^2|<1/\epsilon} \frac{x}{a-x^2}\,\varphi(x)\,dx,$$

with $\varphi \in \mathcal{S}(\mathbb{R})$ defines a tempered distribution. Moreover, prove that if

$$\widehat{f_a}(\xi) = \lim_{\epsilon\downarrow 0}\int\limits_{\epsilon<|a-x^2|<1/\epsilon} e^{-2\pi i(x\cdot\xi)}\,\frac{x}{a-x^2}\,dx,$$

then

$$\|\widehat{f_a}\|_\infty \leq M, \tag{1.47}$$

where the constant M is independent of a.
Hint: Observe that if $a = 0$, $f_a(x)$ is just a multiple of the kernel $1/x$ of the Hilbert transform H. If $a > 0$, then $f_a(x)$ can be written as sum of translations of the kernel of the Hilbert transform H. Since the Hilbert transform satisfies a similar result, (1.47) follows in both cases. (See Example 1.10).

(ii) Show that (1.47) is also satisfied if $a < 0$.
Hint: Use Example 1.6.

1.18 Consider the IVP associated to the wave equation

$$\begin{cases} \partial_t^2 w - \Delta w = 0, \\ w(x,0) = f(x), \\ \partial_t w(x,0) = g(x), \end{cases} \tag{1.48}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Prove that

(i) If $f, g \in C_0^\infty(\mathbb{R}^n)$ are real-valued functions, then using the notation in (1.29), the solution can be described by the following expression:

$$w(x,t) = U'(t)f + U(t)g = \cos(Dt)f + \frac{\sin(Dt)}{D}g, \tag{1.49}$$

with $\widehat{Dh}(\xi) = 2\pi|\xi|\widehat{h}(\xi)$ (see (1.36)).

(ii) If f, g are supported in $\{x \in \mathbb{R}^3 : |x| \le M\}$, show that $w(\cdot,t)$ is supported in $\{x \in \mathbb{R}^3 : |x| \le M + t\}$.

(iii) Assuming $n = 3$ and $f \equiv 0$, prove that

$$w(x,t) = \frac{1}{4\pi t}\int_{\{|y|=t\}} g(x+y)\,dS_y.$$

Hint: Derive and apply the following identity:

$$\int_{\{|x|=t\}} e^{2\pi i\xi\cdot x}\,dS_x = 4\pi t\,\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}.$$

If $g \in C_0^\infty(\mathbb{R}^3)$ is supported in $\{x \in \mathbb{R}^3 : |x| \le M\}$, where is the support of $w(\cdot,t)$?

(iv) Assuming $n = 3$ and $g \equiv 0$, prove that

$$w(x,t) = \frac{1}{4\pi t^2}\int_{\{|y|=t\}} [f(x+y) + \nabla f(x+y)\cdot y]\,dS_y. \tag{1.50}$$

(v) If $E(t) = \int_{\mathbb{R}^n} ((\partial_t w)^2 + |\nabla_x w|^2)(x,t)dx$, then prove that for any $t \in \mathbb{R}$,

$$E(t) = E_0 = \int_{\mathbb{R}^n} (g^2 + |\nabla_x f|^2)(x)\,dx.$$

Hint: Use integration by parts and the equation.

(vi) (Brodsky [Br]) Show that

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} (\partial_t w)^2(x,t)\,dx = \frac{E_0}{2}.$$

Hint: Use the Riemann–Lebesgue lemma (Theorem 1.1(3)).

1.19 Consider the IVP (1.28) with initial data $v_0 \in C_0^\infty(\mathbb{R})$. Prove that for any $t \neq 0$ $v(\cdot,t)$ does not have compact support.