# WELL-POSEDNESS FOR THE INITIAL VALUE PROBLEM ASSOCIATED TO THE ZAKHAROV-KUZNETSOV (ZK) EQUATION IN ASYMMETRIC SPACES.

ÉMILE DELÉAGE[1] AND FELIPE LINARES[2]

Abstract. We study well-posedness for Zakharov-Kuznetsov and modified Zakharov-Kuznetsov equations in asymmetric spaces. In order to do so, we extend a theory initiated by Kato in [Kat83] to higher dimensions $n \geq 2$. As an application, we prove a result concerning dispersive blow-up for the modified Zakharov-Kuznetsov in dimension 2.

## 1. Introduction and main results

The Zakharov-Kuznetsov (ZK) equation was first formally derived by Zakharov and Kuznetsov in [ZK74], as an asymptotic limit of the Euler-Poisson system, in the setting of the "cold plasma" approximation. This equation describes motion of plasma in a uniform magnetic field, in a long wave small-amplitude limit, and can be stated as

$$\partial_t u + \partial_1 \Delta u + u \partial_1 u = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^n. \tag{1}$$

In [LLS12], this asymptotic limit was rigorously justified. In [HK13], this equation was shown to be an asymptotic limit for the Vlasov-Poisson system. In the case $n = 1$, this equation becomes the well-known Korteweg-de Vries (KdV) equation, which describes waves on shallow water surfaces. Thus equation (1) can be seen as a generalisation of the KdV equation in higher dimensions. Note that (1) is not integrable. However, it possesses conserved quantities (cf [Fam95] for instance). These equations belongs to the larger class of nonlinear dispersive equations (see [LP15] for an introduction to the subject). We will focus on the properties of the initial value problem (IVP) associated to (1), that is

$$\begin{cases} \partial_t u + \partial_1 \Delta u + u \partial_1 u = 0 \qquad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ u|_{t=0} = u_0, \end{cases} \tag{2}$$

and to the IVP associated to the generalised Zakharov-Kuznetsov equation which can be written as

$$\begin{cases} \partial_t u + \partial_1 \Delta u + u^k \partial_1 u = 0 \qquad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^n, \\ u|_{t=0} = u_0, \end{cases} \tag{3}$$

---

*Date*: August 29, 2022.
2010 *Mathematics Subject Classification.* Primary 35Q53; Secondary 35B65.
*Key words and phrases.* Zakharov-Kuznetsov, Asymmetric Spaces, Dispersive Blow-up.
The second author was partially supported by CNPq and FAPERJ/Brazil.

where $k \geq 2$. These IVPs were studied by many authors for an initial data $u_0 \in H^s(\mathbb{R}^n)$. In [Fam95], Faminskii showed local well posedness for (2) in dimension 2, in the setting $H^s$, $s \in \mathbb{Z}^+$. Ever since, a lot of advancements have been made. Still in the two dimensional case, Linares and Pastor proved local well-posedness of (3) with $k = 2$ for $s > 3/4$ by using smoothing effects in [LP09]. The Fourier restriction method was also used by Molinet and Pilod in [MP15] and by Grünrock and Herr in [GH14] to extend local well-posedness of (2) to $s > 1/2$.

In dimension 3, Molinet and Pilod [MP15] and Ribaud and Vento [RV12] proved local and global well-posedness for (2) when $s > 1$. We also mention the recent works of Kinoshita [Kin21] and Herr and Kinoshita [HK21] in which well-posedness for (2) was obtained with the Picard iteration method in the best possible setting: $s > -1/4$ in dimension 2 and $s > (n-4)/2$ when $n \geq 3$.

To describe our results, we define the solution of the linear problem associated to the IVPs (2) and (3) by using a group of unitary operators $\{V(t)\}_{t\in\mathbb{R}}$. This group is given explicitely by the formula $V(t)u_0 = \exp(-t\partial_1\Delta)u_0$, or with the Fourier transform by $\widehat{V(t)u_0}(t,\xi) = \exp(it\xi_1|\xi|^2)\widehat{u_0}(\xi)$.

In [Kat83], Kato studied well-posedness of the IVP associated to the KdV equation (dimension 1) for an initial datum in $H^s \cap L^2_b$, where, if $b \in \mathbb{R}^n$,

$$L^2_b = \left\{ f : \mathbb{R}^n \to \mathbb{R}, \int_{\mathbb{R}^n} f^2(x) e^{2b\cdot x} dx < +\infty \right\}. \tag{4}$$

The key property that Kato used for this particular space is that, after pointwise multiplication by $e^{bx}$ where $b > 0$, the unitary group of evolution $V(t)$ becomes parabolic. More precisely, there exists a parabolic semigroup $\{U_b(t)\}$ such that $e^{bx}V(t) = U_b(t)e^{bx}$ (cf section 9 in [Kat83]). Among other results, he proved that the solution $u \in C([0,+\infty), H^s \cap L^2_b)$ of the IVP associated to KdV for an initial datum $\phi \in H^s \cap L^2_b, s \geq 2$ exists and is unique, with the map $\phi \mapsto u$ being continuous in the associated topologies. Furthermore, he proved that the KdV equation possesses a smoothing property for solution with initial data in this space, and in fact the solution $u$ belongs to $C^\infty(\mathbb{R}^*_+ \times \mathbb{R})$.

Here, we generalize these results for the Zakharov-Kuznetsov and modified Zakharov-Kuznetsov equations in dimension $n \geq 2$, by using a similar method. Our first result covers well-posedness in $H^s \cap L^2_b$ for (2) and (3):

**Theorem 1.** *Let $n \geq 2$, $u_0 \in H^{s_0} \cap L^2_b$ for some $s_0 > n/2$ and $b_1 > (\sum_{k=2}^n b_k^2/3)^{1/2}$. Then there is a unique solution to (2) or (3) such that $u \in C([0,\infty); H^{s_0} \cap L^2_b)$ with the map $u_0 \mapsto u$ continuous in the associated topologies. Moreover, $e^{b\cdot x}u \in C((0,\infty), H^s)$ for any $s < s_0 + 2$.*

For solutions of the KdV equation, see [ILP12] and [KF84].

We also extend the smoothing property discovered by Kato in this particular setting. For the Zakharov-Kuznetsov and the generalised Zakharov-Kuznetsov equation, we obtain the following result:

**Theorem 2.** *Let $n \geq 2, k \geq 1, s_0 > n/2$ and $u \in C([0,\infty), H^{s_0})$ be the solution to (2) if $k = 1$ and to (3) if $k \geq 2$. If $u_0 \in H^{s_0} \cap L^2_b$, with $b \in \mathbb{R}^n$ as in Theorem 1, then $e^{b\cdot x}u \in C((0,\infty), H^\infty)$ with the following estimates: for any $T > 0, s \geq 0$ and $\beta > nk/2$,*

$$\|e^{b\cdot x}u(t)\|_{H^s} \leq Ct^{-\beta s/2}, \qquad 0 < t \leq T, \tag{5}$$

*and for every* $\alpha \geq 3\beta$, $\alpha > \beta(1 + kn/2)$,

$$\|(d/dt)^l e^{b\cdot x} u(t)\|_{H^s} \leq C t^{-(\beta s+\alpha l)/2}, \qquad l = 1, 2, 3, \dots \tag{6}$$

Nonlinear dispersive equations are also known to exhibit what is called a dispersive blow-up: a smooth and bounded initial datum with finite energy can result in a solution which develops pointwise singularities in finite time. This focusing phenomenon is caused by the linear operator which possesses an unbounded dispersion speed. In an unbounded domain, it is then possible that infinitely many waves, initially spatially dispersed, come all together at the same point after a finite time, resulting in a blow-up. Bona and Saut initiated the mathematical study of dispersive blow-up for generalized KdV in [BS93]. We mention [LPS17] for an improvement of their result, and [LPD21] for a more recent study.

Dispersive blow-up was also studied for other nonlinear dispersive equations. In [BPSS14], Bona, Ponce, Saut and Sparber studied dispersive blow-up for the nonlinear Schrödinger equation. In [HT16], the pointwise notion of dispersive blow-up is extended in higher dimension $n \geq 2$ to larger sets such as lines or spheres for the nonlinear Schrödinger equation.

As an application of our previous results, we exhibit an example of dispersive blow-up in the setting $n = 2, k = 2$. This example extends Theorem 1.3 from [LPD21].

**Theorem 3.** *Let* $p > 2$ *and* $s \in \mathbb{N}, s \geq 2$. *For any* $t^* \in \mathbb{R}^*$, *there exists* $u_0 \in H^s(\mathbb{R}^2) \cap W^{s,p}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ *such that the corresponding solution of* (3) *with* $k = 2$ *is global in time and satisfies*

(1) $u \in C(\mathbb{R}, H^s(\mathbb{R}^2))$ *and*
(2) $\|u(t^*)\|_{W^{s,p}}$ *blows up on the whole line* $\{0\} \times \mathbb{R}$, *in the following sense: for any* $U \subset \mathbb{R}^2$ *neighborhood of some* $(0, y) \in \{0\} \times \mathbb{R}$, $\|\partial_x^s u(t^*)\|_{L^p(U)} = +\infty$.
(3) *The non linear part of the solution stays bounded, i.e. for every* $t \in \mathbb{R}$, $\|u(t) - V(t)u_0\|_{W^{s,p}} < +\infty$.

Note that dispersive blow-up was initially defined for the $W^{s,\infty}(\mathbb{R}^2)$ norm caused by the highest derivative, cf for instance [BS93]. However, our proof here only works in the setting $W^{s,p}(\mathbb{R}^2)$ with $p < \infty$. In fact, we can almost prove the same theorem in the $W^{s,\infty}(\mathbb{R}^2)$ setting. The missing part is that in the latter case, the smoothing effect on the nonlinear term defined hereafter in the proof is not sufficient to prove that the blow-up is caused by the linear part of the solution (third property of the solution in Theorem 3).

The paper is organised as follows: in section 2, we give the notations and state a set of useful results that we will need. In section 3, we prove some preliminary results concerning the space $L^2_b$. In particular, we show that the linear group of evolution operators $\{V(t)\}$ becomes parabolic after multiplication by an exponential function. Section 4 is devoted to the proof of Theorem 1. In section 5, we prove Theorem 2. Section 3, 4 and 5 are greatly inspired of sections 9, 10 and 11 in [Kat83]. In section 6, we give two examples of linear dispersive blow-up for the group $\{V(t)\}$, and then we prove Theorem 3.

## 2. Notations and some helpful results

Notations: Let $n \in \mathbb{N}^*$. If $x \in \mathbb{R}^n$, we denote $x = (x_1, \dots, x_n)$. If $1 \leq j \leq n$, we denote $\partial_j = \partial/\partial x_j$ the partial derivative relative to $x_j$. We denote the Laplacian operator by $\Delta = \partial_1^2 + \cdots + \partial_n^2$ and the gradient operator $\nabla = (\partial_i)_{1\le i\le n}$. If $s \in \mathbb{R}$, $H^s(\mathbb{R}^n)$ or $H^s$ is the Sobolev space of order $s$, endowed with the norm $\|\cdot\|_{H^s}$. If $1 \le p \le \infty$, we denote $L^p(\mathbb{R}^n)$ or $L^p$ the Lebesgue associated with $p$, endowed with the norm $\|\cdot\|_p$. If $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ we denote $b \cdot x = b_1 x_1 + \cdots + b_n x_n$.

Preliminary results: we use the following propositions to estimate products in Sobolev spaces (see [BH15] and [Tao06], [KP88] respectively):

**Proposition 1.** *Let $s, s_1, s_2 \in \mathbb{R}$ and $n \in \mathbb{N}$.*

- $s \ge 0$
  *If $s \ge 0$, $\min(s_1, s_2) \ge s$ and $s_1 + s_2 - s > n/2$ then there exists a constant $C > 0$ such that for any $(u, v) \in H^{s_1} \times H^{s_2}$ the pointwise product $uv$ belongs to $H^s$ with $\|uv\|_{H^s} \le C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}$.*
- $s < 0$
  *If $s < 0$, $0 > \min(s_1, s_2) \ge s$, $s_1 + s_2 - s > n/2$ and $s_1 + s_2 \ge 0$, then there exists a constant $C > 0$ such that for any $(u, v) \in H^{s_1} \times H^{s_2}$ the pointwise product $uv$ belongs to $H^s$ with $\|uv\|_{H^s} \le C\|u\|_{H^{s_1}}\|v\|_{H^{s_2}}$.*

**Proposition 2.** *If $s \ge 0$ and $f, g \in H^s \cap L^\infty$, then $fg \in H^s$ with*

$$\|fg\|_{H^s} \le C\left(\|f\|_{H^s}\|g\|_\infty + \|f\|_\infty\|g\|_{H^s}\right).$$

The following lemma is an oscillatory integral estimate (see Lemma 2.3 in [LP09]).

**Lemma 1.** *Let $n = 2$. For any $t \ne 0$ and $u_0 \in L^1(\mathbb{R}^2)$, the following estimate holds:*

$$\|V(t)u_0\|_\infty \le |t|^{2/3}\|u_0\|_1.$$

Finally, this nonlinear smoothing effect comes from Proposition 1.4 of [LPD21]:

**Proposition 3.** *Fix $k \ge 2$. Let $v_0 \in H^s(\mathbb{R}^2)$, $s \in \mathbb{N}^*$ and $v \in C([-T, T], H^s(\mathbb{R}^2)$ the local solution of (3). Then*

$$z(t) = \int_0^t V(t - t')v^k \partial_1 v(t')dt' \in C([-T, T], H^{s+1}(\mathbb{R}^2).$$

## 3. The space $L^2_b$

Here we follow the proof of Kato in [Kat83] for the dimension 1 and try to adapt it for higher dimensions.

The proof of Kato is based on the following commutation property: for $f \in \mathcal{D}', e^{b\cdot x}\partial_i f = (\partial_i - b_i)e^{b\cdot x} f$. Hence, in dimension 1, the operator $\partial_1^3$ of (1) becomes parabolic when $b > 0$. The following lemma generalizes this property in dimension $n$.

**Lemma 2.** *Let $b \in \mathbb{R}^n$ such that $b_1 > 0$ and $\sum_{k=2}^n b_k^2 < 3b_1^2$. Define the semigroup*

$$U_b(t) = \exp\left[-t(\partial_1 - b_1)\sum_{k=1}^n (\partial_k - b_k)^2\right], \qquad t \ge 0. \tag{7}$$

*Then $\{U_b(t) : t > 0\}$ is an infinitely differentiable semigroup on $H^s$ for each $s \in \mathbb{R}$, with*

$$\|\partial^\alpha U_b(t)\| \le C_\alpha t^{-|\alpha|/2}\exp(b_1|b|^2 t), \qquad \alpha \in \mathbb{N}^n, \tag{8}$$

$$\|(d/dt)U_b(t)\| \le C(t^{-3/2} + 1)\exp(b_1|b|^2 t). \tag{9}$$

*$U_b(t)$ is bounded from $H^s$ to $H^{s'}$, with*

$$\|U_b(t)\|_{B(H^s,H^{s'})} \leq C(t^{-(s'-s)/2}+1)\exp(b_1|b|^2 t), \qquad s \leq s'. \tag{10}$$

*Proof.* In Fourier, $U_b(t)$ acts like a multiplication by the factor

$$\lambda(t,\xi) = \exp\left[-t(i\xi_1 - b_1)\sum_{k=1}^{n}(i\xi_k - b_k)^2\right].$$

Developing the products gives

$$|\lambda(t,\xi)| = e^{b_1|b|^2 t}\exp[-t\theta(\xi)]$$

with $\theta(\xi) = 2\xi_1\xi\cdot b + b_1|\xi|^2$. Denote $\tilde{\xi} = (\xi_2\ldots,\xi_n)$ and $\tilde{b} = (b_2,\ldots,b_n)$. For every $b_1 > \epsilon > 0$, we get using Cauchy-Schwarz inequality

$$\begin{aligned}\theta(\xi) &= 3b_1\xi_1^2 + b_1|\tilde{\xi}|^2 + 2\xi_1\tilde{\xi}\cdot\tilde{b} \geq 3b_1\xi_1^2 + b_1|\tilde{\xi}|^2 - 2|\tilde{\xi}||\tilde{b}||\xi_1| \\ &\geq 3b_1\xi_1^2 + b_1|\tilde{\xi}|^2 - (b_1-\epsilon)|\tilde{\xi}|^2 - \xi_1^2|\tilde{b}|^2/(b_1-\epsilon) \\ &= \epsilon b_1|\tilde{\xi}|^2 + \frac{3b_1^2 - |\tilde{b}|^2 - 3b_1\epsilon}{b_1-\epsilon}\xi_1^2.\end{aligned}$$

By choosing $0 < 3b_1\epsilon < 3b_1^2 - |\tilde{b}|^2$, we obtain that there is $C > 0$ such that $\theta(\xi) \geq C|\xi|^2$. Hence the semigroup $U_b$ is parabolic and the results follow. □

**Lemma 3.** *Let*

$$e^{b\cdot x}u \in L^\infty([0,T];L^2), \qquad e^{b\cdot x}f \in L^\infty([0,T],H^{-1}), \tag{11}$$

$$du/dt + \partial_1\Delta u = f, \qquad 0 < t < T. \tag{12}$$

*Then one has*

$$e^{b\cdot x}u \in C([0,T],L^2)\cap C((0,T],H^s) \qquad \textit{for every } s<1, \tag{13}$$

$$e^{b\cdot x}u(t) = U_b(t)e^{b\cdot x}u(0) + \int_0^t U_b(t-t')e^{b\cdot x}f(t')dt'. \tag{14}$$

*and the following estimate, for $0 < s < 1$ and $0 < t \leq T$:*

$$\|e^{b\cdot x}u\|_{H^s} \leq Ct^{-s/2}\left(\|e^{b\cdot x}u\|_{L^\infty([0,T],L^2)} + \|e^{b\cdot x}f\|_{L^\infty([0,T],H^{-1})}\right).$$

*Proof.* By multiplying the equation by $e^{b\cdot x}$ and using the commutation property, we obtain

$$(d/dt)e^{b\cdot x}u + (\partial_1 - b_1)\sum_{k=1}^{n}(\partial_k - b_k)^2 e^{b\cdot x}u = e^{b\cdot x}f. \tag{15}$$

This gives the integral form of the equation. We can then use Lemma 2 to obtain that $\|U_b(t-t')\|_{B(H^{-1},H^s)} \leq C(1+(t-t')^{(-1-s)/2})$, which is integrable if $s<1$. We can then bound the nonlinear integral term by

$$\|\int_0^t U_b(t-t')e^{b\cdot x}f(t')dt'\|_s \leq CT + C\int_0^t (t-t')^{(-1-s)/2}dt'$$

and the change of variables $r = t'/t$ shows that the last integral is bounded on $[0,T]$ as a function of $t$. The estimate follows. □

**Lemma 4.** *Let $n \geq 2$, $T > 0$ and*

$$du/dt + \partial_1 \Delta u + a(t)\partial_1 u = 0, \qquad 0 \leq t \leq T, \tag{16}$$

*where $a \in C([0,T], H^{s_0})$ for some $s_0 > n/2$. If $e^{b\cdot x}u \in L^\infty([0,T], L^2)$, then $e^{b\cdot x}u \in C((0,T], H^s)$ for any $s < s_0 + 2$, with the estimate*

$$\|e^{b\cdot x}u\|_{H^s} \leq C(\|e^{b\cdot x}u\|_{L^\infty([0,T],L^2)}, \|a\|_{L^\infty([0,T],H^{s_0})})t^{-s'/2}, \qquad 0 < t \leq T.$$

*Proof.* We want to apply the previous lemma, with $f = -a(t)\partial_1 u$. Since $e^{b\cdot x}u \in L^\infty([0,T], L^2)$, we get that $e^{b\cdot x}\partial_1 u \in L^\infty([0,T], H^{-1})$, hence by Proposition 1, since $s_0 > n/2$, $e^{b\cdot x}f = -a(t)e^{b\cdot x}\partial_1 u \in L^\infty([0,T], H^{-1})$. The previous lemma then gives that $e^{b\cdot x}u \in C((0,T], H^s)$ for every $s < 1$ with

$$e^{b\cdot x}u(t) = U_b(t)e^{b\cdot x}u(0) - \int_0^t U_b(t-t')e^{b\cdot x}a(t')\partial_1 u(t')dt'$$

and we obtain the estimate for every $s < 1$. Now we fix $s \in [1/2, s_0 + 3/2)$ and show that if the result is true for $s - 1/2$, then it is also true for $s$. We first note that $t^{s/2}u$ solves

$$\frac{d}{dt}t^{s/2}u - \frac{s}{2}t^{s/2-1}u + \partial_1 \Delta t^{s/2}u + t^{s/2}a(t)\partial_1 u = 0.$$

Hence we obtain the following integral equation, for $0 < t \leq T$ (note that the initial value of $t^{s/2}e^{b\cdot x}u(t)$ is zero since $e^{b\cdot x}u(0) \in L^2$):

$$t^{s/2}e^{b\cdot x}u(t) = \int_0^t U_b(t-t')\left[\frac{s}{2}(t')^{s/2-1}e^{b\cdot x}u(t') - (t')^{s/2}a(t')e^{b\cdot x}\partial_1 u(t')\right]dt'.$$

Hence by using the properties of the semigroup $U_b$ we obtain

$$\begin{aligned} t^{s/2}\|e^{b\cdot x}u\|_{H^s} \leq & C\int_0^t \Big[(t-t')^{-1/4}(t')^{s/2-1}\|e^{b\cdot x}u\|_{H^{s-1/2}} \\ & + (t-t')^{-3/4}(t')^{s_0/2}\|a(t')e^{b\cdot x}\partial_1 u(t')\|_{H^{s-3/2}}\Big]dt'. \end{aligned}$$

Now by hypothesis the first term can be estimated by

$$(t-t')^{-1/4}(t')^{s/2-1}\|e^{b\cdot x}u\|_{H^{s-1/2}} \leq C(t-t')^{-1/4}(t')^{-3/4}$$

which is integrable, with an integral bounded for $t \in [0,T]$, and the second one by

$$\begin{aligned} & (t-t')^{-3/4}(t')^{s/2}\|a(t')e^{b\cdot x}\partial_1 u(t')\|_{H^{s-3/2}} \\ \leq & C(t-t')^{-3/4}(t')^{s/2}\|a(t')\|_{H^s}\|e^{b\cdot x}\partial_1 u\|_{H^{s-3/2}} \\ \leq & C(t-t')^{-3/4}(t')^{s/2}\|e^{b\cdot x}u\|_{H^{s-1/2}} \leq C(t-t')^{-3/4}(t')^{1/4} \end{aligned}$$

which is also integrable, with a bounded integral. Here we used again Proposition 1 and the hypothesis on $\|e^{b\cdot x}u\|_{H^{s-1/2}}$. This concludes the proof. □

## 4. Proof of Theorem 1

Let $s_0 > n/2$ and $b \in \mathbb{R}^n$ such that $b_1 > (\sum_{k=2}^n b_k^2/3)^{1/2}$. In this section, we show well-posedness of (2) and (3) in $H^{s_0} \cap L_b^2$. As a consequence of the well posedness theory in $H^{s_0}$, we already know that there exists a unique solution in $C([0,T], H^{s_0})$, which is global if the norm of the initial data is sufficiently small. To simplify computations, we will restrict ourselves to the setting of global solutions, but the results stay true in the general case. Here, it is enough to show that $e^{b\cdot x}u(t) \in L^2$. In the following, we fix $k \geq 1$.

Again, we follow Kato and introduce the bounded weight functions

$$q(x) = \frac{e^{b\cdot x}}{(1+\epsilon e^{2b\cdot x})^{1/2}}, \qquad r(x) = \frac{e^{b\cdot x}}{1+\epsilon e^{2b\cdot x}}, \qquad p(x) = q(x)^2 \tag{17}$$

depending on a parameter $\epsilon > 0$. Both $q$ and $r$ are $L^\infty$ functions with the $L^\infty$ norm proportional to $\epsilon^{1/2}$, and both tend monotonically to $e^{b\cdot x}$ as $\epsilon \downarrow 0$. We note several properties of these functions required in the sequel:

$$\partial_i p = 2b_i r^2, \qquad |\partial_i\partial_j p| < 4|b_i b_j| r^2, \qquad |\partial_i\partial_j\partial_k p| < 8|b_i b_j b_k| r^2, \qquad |\partial_i r| < |b_i| r. \tag{18}$$

We now take $u$ the solution of the problem in $H^s$, multiply equation (1) (or (3)) by $2pu$ and integrate over $\mathbb{R}^n$ to obtain

$$\frac{d}{dt}\int pu^2 = -2\int pu(\partial_1\Delta u + u^k\partial_1 u). \tag{19}$$

Integrations by parts give that

$$\int pu^{k+1}\partial_1 u = -\frac{1}{k+2}\int (\partial_1 p)u^{k+2} \tag{20}$$

and

$$\int pu\partial_1\Delta u = -\frac{1}{2}\int (\partial_1\Delta p)u^2 - 2(\partial_1 u)\nabla p\cdot\nabla u - (\partial_1 p)|\nabla u|^2. \tag{21}$$

Now, using (20) leads to

$$\int -2(\partial_1 u)\nabla p\cdot\nabla u - (\partial_1 p)|\nabla u|^2 = -2\int r^2[b_1|\nabla u|^2 + 2(\partial_1 u)b\cdot\nabla u]. \tag{22}$$

Note that $b_1|\nabla u|^2 + 2(\partial_1 u)b\cdot\nabla u = \theta(\nabla u) \geq C|\nabla u|^2$, where $C > 0$, in virtue of the condition on $b$ (see the proof of Lemma 2 for the definition and properties of $\theta$). Using again (20) and putting everything together yields

$$\frac{d}{dt}\int pu^2 < 8b_1|b|^2\int r^2u^2 + \frac{4}{k+2}b_1\int r^2u^{k+2} - C\int r^2|\nabla u|^2. \tag{23}$$

Since $u \in H^{s_0}$ with $s_0 > n/2$, $u \in L^\infty$. Finally, we get

$$\frac{d}{dt}\|qu\|_2^2 \leq K(\|u\|_{H^{s_0}})\|ru\|_2^2 \leq K(\|u\|_{H^{s_0}})\|qu\|_2^2. \tag{24}$$

Since $K$ is independent of $\epsilon$, going to the limit $\epsilon \downarrow 0$ gives

$$\|e^{b\cdot x}u(t)\|_2^2 \leq e^{Kt}\|u_0\|_2^2, \qquad 0 \leq t \leq T \tag{25}$$

with $K = K(\|u\|_{L^\infty([0,T],H^{s_0})})$. Since $a(t) = u \in C([0,T],H^{s_0})$, we can apply Lemma 4 to obtain that $e^{b\cdot x}u \in C([0,T],L^2) \cap C((0,T],H^s)$ for any $s \leq s_0 + 2$. Thus we have proved the main part of Theorem 1.

It remains to prove the continuous dependence $u_0 \mapsto u$. Since this is known for the $H^{s_0}$ norm by the $H^{s_0}$ theory, it suffices to show that the map $e^{b\cdot x}u_0 \mapsto e^{b\cdot x}u$ is continuous in the $L^2$ norms, uniformly in $t \in [0,T]$. This can be seen by the following integral equation satisfied by $v(t) = e^{b\cdot x}u(t)$:

$$v(t) = U_b(t)v_0 - \int_0^t W(t,t')v(t')dt', \tag{26}$$

where

$$W(t,t') = (\partial_1 - b_1)U_b(t-t')u^k(t')/(k+1)$$

is an operator valued kernel such that $\|W(t,t')\|_{B(L^2)} \leq C(t-t')^{-1/2}$ (because $u \in H^{s_0}$ and $s_0 > n/2$). This equation is obtained by lemma 1 with $f = -u^k\partial_1 u$. It should be noted that $W(t,t')$ depends on $u$ and hence on $u_0$, but the dependence is known to be continuous in the $H^{s_0}$ norm which is weaker than the $H^{s_0} \cap L^2_b$ norm.

## 5. Proof of Theorem 2

We present here the proof of Theorem 2.

*Proof.* We start by proving the first inequality. By Theorem 1 and the estimate of Lemma 4, we already know that it is true for any $s < s_0+2$ (note that $\beta > nk/2 \geq 1$, hence the estimate of Lemma 4 for $s < s_0+2$ is stronger than the one that we need to prove). Now we fix $\delta > 0$ and show that if the estimate holds for some $s-\delta$ with $s \geq 1/2$, then it also holds for $s$. We write again the integral equation satisfied by $t^{\beta s/2}e^{b\cdot x}u(t)$:

$$t^{\beta s/2}e^{b\cdot x}u(t) = \int_0^t U_b(t-t')\left[\frac{\beta s}{2}(t')^{\beta s/2-1}e^{b\cdot x}u(t') - (t')^{\beta s/2}u(t')^k e^{b\cdot x}\partial_1 u(t')\right]dt'.$$

Hence by using the properties of the semigroup $U_b$ we obtain

$$\begin{aligned} t^{\beta s/2}\|e^{b\cdot x}u\|_{H^s} \leq & C\int_0^t \Big[(t-t')^{-\delta/2}(t')^{\beta s/2-1}\|e^{b\cdot x}u\|_{H^{s-\delta}} \\ & +(t-t')^{-(\delta+1)/2}(t')^{\beta s/2}\|u(t')^k e^{b\cdot x}\partial_1 u(t')\|_{H^{s-1-\delta}}\Big]\,dt'. \end{aligned}$$

Now by hypothesis the first term can be estimated by

$$(t-t')^{-\delta/2}(t')^{\beta s/2-1}\|e^{b\cdot x}u\|_{H^{s-\delta}} \leq C(t-t')^{-\delta/2}(t')^{\beta\delta/2-1},$$

and the integral of this term is finite and bounded in $t \leq T$ whenever $\beta \geq 1$ and $0 < \delta < 2$ (to prove this, one can again make the change of variables $r = t'/t$).

For the second term, we write that

$$\|u(t')^k e^{b\cdot x}\partial_1 u(t')\|_{H^{s-\delta-1}} = \frac{1}{k+1}\|(\partial_1 - b_1)e^{b\cdot x}u(t')^{k+1}\|_{H^{s-\delta-1}} \leq C\|e^{b\cdot x}u(t')^{k+1}\|_{H^{s-\delta}}.$$

To estimate the norm of $e^{b\cdot x}u^{k+1} = (e^{b\cdot x/(k+1)}u)^{k+1}$ we use Proposition 2. By induction on $k$, we obtain the following generalised version: for any $v \in H^{s'} \cap L^\infty$ and $k \geq 0$,

$$\|v^{k+1}\|_{H^{s'}} \leq C\|v\|_\infty^k\|v\|_{H^{s'}}.$$

We use this last inequality with $v = e^{b\cdot x/(k+1)}u$ and $s' = s-\delta$, combined with the Sobolev embedding theorem, to obtain

$$\begin{aligned} \|e^{b\cdot x}u(t')^{k+1}\|_{H^{s-\delta}} & \leq C\|e^{b\cdot x/(k+1)}u(t')\|_{H^{s-\delta}}\|e^{b\cdot x/(k+1)}u(t')\|_\infty^k \\ & \leq C\|e^{b\cdot x/(k+1)}u(t')\|_{H^{s-\delta}}\|e^{b\cdot x/(k+1)}u(t')\|_{H^{s_1}}^k, \end{aligned}$$

with $s_1 > n/2$. Now we use the hypothesis for $s-\delta$, and the estimate of Lemma 4 for $s_1$, to obtain that

$$(t-t')^{-(\delta+1)/2}(t')^{\beta s/2}\|u(t')^k e^{b\cdot x}\partial_1 u(t')\|_{H^{s-1-\delta}} \leq C(t-t')^{-(\delta+1)/2}(t')^{(\beta\delta-ks_1)/2}.$$

The integral of this expression is finite and bounded in $t \leq T$ if we take $1 > \delta > 0$ and $s_1 > n/2$ such that $\beta \geq 1 + (ks_1-1)/\delta$ (once again, this bound comes from the change of variables $r = t'/t$). The hypothesis $\beta > nk/2$ ensures that we can find such $s_1$ and $\delta$.

Note that we are allowed to use the property for $e^{b\cdot x/(k+1)}u$ instead of $e^{b\cdot x}u$ because $e^{b\cdot x/(k+1)}u_0 \in L^2$, with $\|e^{b\cdot x/(k+1)}u_0\|_2^2 \leq \|u_0\|_2^{1-1/(k+1)}\|e^{b\cdot x}u_0\|_2^{1/(k+1)}$ (Hölder). The homogeneous condition verified by $b$ is also true for $b/(k+1)$, hence we can use the hypothesis for $e^{b\cdot x/(k+1)}u$ instead of $e^{b\cdot x}u$. Hence the decay (5) is valid for every $s \geq 0$.

Now we prove the second inequality by induction on $l$. For $l = 0$, it is known by (5). Assuming that it has been proved for all $s \geq 0$ up to a given $l$, we prove it for $l+1$. Again using (17) with $f = -u^k\partial_1 u$, we obtain on differentiation

$$(d/dt)^{l+1}e^{b\cdot x}u = -(\partial_1 - b_1)\sum_{k=1}^{n}(\partial_k - b_k)^2(d/dt)^l e^{b\cdot x}u - (d/dt)^l e^{b\cdot x}u^k\partial_1 u. \tag{27}$$

The $H^s$ norm of the first term on the right is dominated by

$$\|(d/dt)^k e^{b\cdot x}u\|_{H^{s+3}} \leq Ct^{-(\beta(s+3)+\alpha l)/2} \leq Ct^{-(\beta s+\alpha(l+1))/2}$$

by induction hypothesis. This gives the required estimate.

For the second term in (29), we have as above

$$\begin{aligned}\|(d/dt)^l e^{b\cdot x}u^k\partial_1 u\|_{H^s} \leq& C\|(d/dt)^l e^{b\cdot x}u^{k+1}\|_{H^{s+1}}\\ \leq& C\sum_{l_1+\cdots+l_{k+1}=l}\|(d^{l_1}v)\dots(d^{l_{k+1}}v)\|_{H^{s+1}},\end{aligned}$$

where we have written $v = e^{b\cdot x/(k+1)}u$ and $d = d/dt$ for simplicity. Using Proposition 2 multiple times again, we obtain

$$\|(d^{l_1}v)\dots(d^{l_{k+1}}v)\|_{H^{s+1}} \leq C\sum_{i=1}^{k+1}\|d^{l_i}v\|_{H^{s+1}}\prod_{j\neq i}\|d^{l_j}v\|_\infty.$$

By induction hypothesis and $H^{s_1} \hookrightarrow L^\infty$, where $s_1 = (\alpha/\beta - 1)/k$ (since we know that $\alpha > \beta(kn/2+1)$), this is dominated by $Ct^{-m/2}$, where

$$m = \beta(s+1) + \alpha l_i + \sum_{j\neq i}(\beta s_1 + \alpha l_j) = \beta(s+1) + \alpha l + k\beta s_1 = \beta s + (l+1)\alpha.$$

This is the required estimate. □

## 6. Application: dispersive blow-up in dimension $n \geq 2$

**6.1. Linear dispersive blow-up.** In this section, we construct an initial datum for the linear problem associated to (1) such that the linear evolution exhibits a singularity at a given time, on a linear subspace of dimension $d < n$. More precisely, we state the following

**Proposition 4.** *Let $n \geq 2$ and $1 \leq d \leq n$. For $t \in \mathbb{R}$, let $V(t) = e^{-t\partial_1\Delta}$. Recall that $(V(t))_{t\in\mathbb{R}}$ is a group of evolution operators that preserves $H^s$ norms. For any $t^* \in \mathbb{R}^*$, there exists $u_0 \in H^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that:*

(1) *For every $t \in \mathbb{R} - \{t^*\}$, $u(t) = V(t)u_0 \in H^1(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.*
(2) *$u(t,x) \in C^0(\mathbb{R}\times\mathbb{R}^n)$.*
(3) *$|Du(t,x)| \to +\infty$ when $(t,x) \to (t^*,x^*)$, for every $x^* \in V_d$, where*

$$V_d = \{(x_1,\dots,x_n) \in \mathbb{R}^n, x_1 = \cdots = x_d = 0\}.$$

*Proof.* For $x \in \mathbb{R}^n$, let us write $x = (y,z)$ where $y = (x_1, \dots, x_d)$ and $z = (x_{d+1}, \dots, x_n)$. Let $\phi(x) = e^{-2\pi|y|}e^{-\pi|z|^2}$. Note that $\phi$ has an exponential decay, which will enable to use the smoothing properties of Lemma 2. Take any $b \in \mathbb{R}^n$ such that $1 \geq b_1 > 0$ and $\sum_{k=2}^n b_k^2 < 3b_1^2$. Then $e^{b\cdot x}\phi$ belongs to $L^2(\mathbb{R}^n)$. Note that $e^{b\cdot x}V(t)\phi = U_b(t)e^{b\cdot x}\phi$, where $U_b(t)$ is defined as in Lemma 2. By the smoothing properties of $U_b(t)$ stated in Lemma 2, for any $t > 0$, the function $U_b(t)e^{b\cdot x}\phi$ belong to $H^\infty(\mathbb{R}^n)$, hence is smooth. Thus $V(t)\phi$ is also a smooth function.

For negative times, use the fact that $e^{-b\cdot x}\phi$ also belongs to $L^2(\mathbb{R}^n)$, and

$$e^{-b\cdot x}V(-t)\phi = U_{-b}(-t)e^{-b\cdot x}\phi.$$

Reversing the proof of Lemma 2 shows that $U_{-b}$ is parabolic backwards in time. Hence here again $e^{-b\cdot x}V(-t)\phi$ and then $V(-t)\phi$ are smooth functions, for any $t > 0$.

The candidate for Proposition 4 is thus $u_0 = V(-t^*)\phi$. By the previous arguments, $V(t)u_0$ is smooth for any $t \neq t^*$. We then show that $u_0 \in H^1(\mathbb{R}^n)$. By the properties of $V(t)$, it is enough to check that $\phi \in H^1(\mathbb{R}^n)$. The Fourier transform of $\phi$ is given by

$$\hat{\phi}(\xi_y, \xi_z) = C\frac{e^{-\pi|\xi_z|^2}}{(1+|\xi_y|^2)^{(d+1)/2}} := \hat{f}(\xi_y)\hat{g}(\xi_z)$$

where $C > 0$ is a constant. Since $g \in S(\mathbb{R}^{n-d})$ and $f \in H^{1+d/2-}(\mathbb{R}^d)$, $\phi \in H^{1+d/2-}(\mathbb{R}^n)$. Note that, for any $s \geq 1$,

$$|D_y^s\phi(0,z)| = C\,e^{-\pi|z|^2}\int_{\mathbb{R}^d}\frac{|\xi_y|^s}{(1+|\xi_y|^2)^{\frac{d+1}{2}}}\,d\xi_y = \infty.$$

□

We also state the following example in the case $n = 2$:

**Proposition 5.** *Let $s \in \mathbb{N}^*$. For any $t^* \in \mathbb{R}^*$, there exists $u_0 \in H^s(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ such that:*

(1) *For every $t \in \mathbb{R} - \{t^*\}$, $u(t) = V(t)u_0 \in H^s(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$.*
(2) *$u(t,x) \in C^{s-1}(\mathbb{R} \times \mathbb{R}^2)$.*
(3) *$|\partial_x^s u(t,x)| \to +\infty$ when $(t,x) \to (t^*, x^*)$, $\forall x^* \in \{(x,y) \in \mathbb{R}^2, x = 0\}$.*

*Proof.* Let $p > 2$ and $\phi_p(x,y) = |x|^{s-1/p}e^{-x^2-y^2}$. Then $\phi_p \in H^s \cap L_b^2$ for any $b \in \mathbb{R}^2$. The proof of the previous proposition shows that $V(t)\phi_p$ is smooth for any $t \neq 0$. Note that $\partial_x^s\phi_p(x,y) = Csgn(x)^s|x|^{-1/p}e^{-x^2-y^2} + g(x,y)$, where $g$ is a continuous function with exponential decay. In particular, $\partial_x^s\phi_p \in L^2$ and $|\partial_x^s\phi_p(x,y)| \to \infty$ for any $y$ when $x$ goes to zero. Taking $u_0 = V(-t^*)\phi_p$ again enables to end the proof. □

6.2. **Non linear dispersive blow-up on a line.** We give here the proof of Theorem 4.

*Proof.* We use here a proof very similar to the one of Theorem 1.3 in [LPD21]. Consider $\phi_p$ as in the proof of Proposition 5 and define $u_0 = V(-t^*)\phi_p$. We write

$$u(t) = V(t)u_0 + \int_0^t V(t-t')u^2\partial_x u(t')dt' := V(t)u_0 + z(t)$$

the solution of (3) with $n = k = 2$. Since $\{V(t)\}_{t\in\mathbb{R}}$ is an unitary group in $H^s$ and $\phi_p \in H^s(\mathbb{R}^2)$, $u_0 \in H^s(\mathbb{R}^2)$. Up to multiplying $\phi_p$ by a small constant, we can suppose that $u(t)$ is globally defined and $u \in C(\mathbb{R}, H^s)$. By the Proposition 1.4 of [LPD21] (cf Proposition 3), $z(t) \in H^{s+1}(\mathbb{R}^2) \subset W^{s,p}(\mathbb{R}^2)$ for all times. By the proof of Proposition 5, $V(t^*)u_0 = \phi_p \in W^{s,1}(\mathbb{R}^2)$. By Lemma 1, for any $t \neq t^*$,

$$\|V(t)u_0\|_{W^{s,\infty}(\mathbb{R}^2)} \leq C|t-t^*|^{-2/3}\|\phi_p\|_{W^{s,1}(\mathbb{R}^2)}.$$

Hence for any $t \neq t^*$, $V(t)u_0 \in W^{s,\infty}(\mathbb{R}^2) \cap H^s(\mathbb{R}^2) \subset W^{s,p}(\mathbb{R}^2)$. Hence the solution $u(t) = V(t)u_0 + z(t)$ belongs to $W^{s,p}$ whenever $t \neq t^*$. Finally, by Proposition 5, $u_0 \in C^\infty(\mathbb{R}^2)$.

But for any $y \in \mathbb{R}$, $|\partial_x^s \phi_p(x,y)| \sim C|x|^{-1/p} e^{-x^2-y^2}$ as $x$ goes to zero. Hence the $L^p$ norm of $\partial_x^s u(t^*) = \partial_x^s \phi_p + \partial_x^s z(t^*)$ blows up on any open subset $U \subset \mathbb{R}^2$ such that $U \cap (\{0\} \times \mathbb{R}) \neq \emptyset$. $\square$

## References


[BH15] A. Behzadan and M. Holst. Multiplication in Sobolev spaces, revisited. *Arkiv för Matematik*, 59:275–306, 2015.

[BPSS14] J.L. Bona, G. Ponce, J.-C. Saut, and C. Sparber. Dispersive blow-up for nonlinear Schrödinger equations revisited. *Journal de Mathématiques Pures et Appliquées*, 102(4):782–811, 2014.

[BS93] J.L. Bona and J.C. Saut. Dispersive blowup of solutions of generalized Korteweg-de Vries equations. *Journal of Differential Equations*, 103(1):3–57, 1993.

[Fam95] A. Faminskii. The Cauchy problem for the Zakharov-Kuznetsov equation. *Differential Equations*, 31:1002–1012, 1995.

[GH14] A. Grünrock and S. Herr. The Fourier restriction norm method for the Zakharov-Kuznetsov equation. *Discrete and Continuous Dynamical Systems - A*, 34(5):2061–2068, 2014.

[HK13] D. Han-Kwan. From Vlasov–Poisson to Korteweg–de Vries and Zakharov–Kuznetsov. *Communications in Mathematical Physics*, 324(3):961–993, 2013.

[HK21] S. Herr and S. Kinoshita. The Zakharov–Kuznetsov equation in high dimensions: small initial data of critical regularity. *Journal of Evolution Equations*, 21(2):2105–2121, 2021.

[HT16] Y. Hong and M. Tasković. On dispersive blow-ups for the nonlinear Schrödinger equation. *Differential and Integral Equations*, 29(9/10):875 – 888, 2016.

[ILP12] P. Isaza, F. Linares, and G. Ponce. On decay properties of solutions of the k-generalized KdV equation. *Communications in Mathematical Physics*, 324:129–146, 2012.

[Kat83] T. Kato. On the Cauchy problem for the (generalized) Korteweg-de Vries equation. *Advances in Mathematics Supplementary Studies*, 8:93–128, 1983.

[KF84] S. N. Kruzhkov and A. V. Faminskiĭ. Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation. *Mathematics of the USSR-Sbornik*, 48(2):391–421, 1984.

[Kin21] S. Kinoshita. Global well-posedness for the Cauchy problem of the Zakharov-kuznetsov equation in 2d. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 38(2):451–505, 2021.

[KP88] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Communications on Pure and Applied Mathematics*, 41:891–907, 1988.

[LLS12] D. Lannes, F. Linares, and J.-C. Saut. The Cauchy problem for the Euler-Poisson system and derivation of the Zakharov-Kuznetsov equation. *Progress in Nonlinear Differential Equations and Their Application*, 84:181–213, 2012.

[LP09] F. Linares and A. Pastor. Well-posedness for the two-dimensional modified Zakharov–Kuznetsov equation. *SIAM J. Math. Analysis*, 41:1323–1339, 01 2009.

[LP15] F. Linares and G. Ponce. *Introduction to Nonlinear Dispersive Equations*. Universitext. Springer New York, 2015.

[LPD21] F. Linares, A. Pastor, and J. Drumond Silva. Dispersive blow-up for solutions of the zakharov-kuznetsov equation. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 38(2):281–300, 2021.

[LPS17] F. Linares, G. Ponce, and D.L. Smith. On the regularity of solutions to a class of nonlinear dispersive equations. *Mathematische Annalen*, 369:797–837, 2017.

[MP15] L. Molinet and D. Pilod. Bilinear strichartz estimates for the Zakharov–Kuznetsov equation and applications. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 32(2):347–371, 2015.

[RV12] F. Ribaud and S. Vento. Well-posedness results for the three-dimensional Zakharov–Kuznetsov equation. *SIAM Journal on Mathematical Analysis*, 44(4):2289–2304, 2012.

[Tao06] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. Conference Board of the Mathematical Sciences regional conference series in mathematics. American Mathematical Society, 2006.

[ZK74] V. Zakharov and E.A. Kuznetsov. Three-dimensional solitons. *Soviet Physics JETP*, 29:594–597, 1974.

[1]ENS de Lyon, 35 parvis René Descartes, 69007 Lyon, France
*Email address*: emile.deleage@ens-lyon.fr

[2]IMPA, Estrada Dona Castorina 110, Rio de Janeiro 22460-320, RJ, Brazil
*Email address*: linares@impa.br