Multiple Scale Asymptotics of Fast Mean Reversion Stochastic Volatility Models

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1 Introduction

Classical Black-Scholes models are plagued by some well known pitfalls. To cite a few: Log-normality of asset prices is not verified by statistical tests, option prices are subject to the smile effects, and volatility of the prices tends fluctuate with time and revert to a mean value.

One of the ways to attack these problems is to consider *stochastic volatility models*. More specifically, we consider the following risky asset dynamics

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \qquad \sigma_t = f(Y_t) \qquad dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$$

where $\widehat{Z}_t$ is a linear combination of two independent Brownian motions (W_t) and (Z_t) with correlation ρ. Furthermore, m is the mean for the hidden process, α is the rate of mean-reversion, β is the volatility of the volatility (vol-vol).

Following the seminal work (Fouque, Papanicolaou, & Sircar 2000), we consider the asymptotic behavior of option prices for stochastic volatility models for which the volatility process fluctuates on a much faster time scale than that defined by the risk-less interest rate. However, a crucial assumption in (Cotton, Fouque, Papanicolaou, & Sircar 2004; Fouque, Papanicolaou, & Sircar 2000; Fouque, Papanicolaou, & Sircar 2001b; Fouque, Papanicolaou, & Sircar 2001a; Fouque, Papanicolaou, Sircar, & Solna 2003; Cotton, Fouque, Papanicolaou, & Sircar 2004; Howison, Rafailidis, & Rasmussen 2004) is that the variance of the volatility process is of order one. On the other hand, statistical studies of available data, already performed in (Fouque, Papanicolaou, & Sircar 2000), suggest the possibility that such variance could be somewhat smaller than order one. This is consistent with statistical studies we performed on the Brazilian stock market data (IBOVESPA), and presented in section 3. Instead of dwelling upon the range of validity for the asymptotic results in (Fouque, Papanicolaou, & Sircar 2000) as far as actual market data is concerned, we explore other asymptotic regimes. This analysis of small vol-vol regimes naturally leads us to perform a singular pertubation analysis through multiple scales. For background on perturbation methods see, for example, (Hinch 1991).

We derive the corresponding asymptotic formulae for option prices using a multiple scales approach, and find that the first order correction displays a dependence both on a fast time scale and on the hidden state. It turns out that the fast time scale play an important role in rendering the asymptotic expansion for the implied volatility bounded.

In the stochastic volatility context the price $P(t,x,y)$ satisfies:

$$\frac{\partial P}{\partial t}+\frac{1}{2}f(y)^2x^2\frac{\partial^2 P}{\partial x^2}+\rho\beta x f(y)\frac{\partial^2 P}{\partial x\partial y}+$$
$$\frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2}+r(x\frac{\partial P}{\partial x}-P)+(\alpha(m-y)-\beta\Lambda(t,x,y))\frac{\partial P}{\partial y}=0$$

with final condition: $P(T,x,y)=h(x)$. Here, $\Lambda(t,x,y)$ is associated to the so called market price of volatility risk.

2 Different Asymptotic Regimes

The first step in our approach is to consider the *dimensionless* pricing equation. More precisely, we work with the adimensional variables: $\widehat{t}$=rt, $\widehat{f}=\sqrt{r}f$, $\widehat{x}=x/K$, and drop the hats altogether.

$$\frac{\partial P}{\partial t}+\frac{1}{2}(f(y))^2x^2\frac{\partial^2 P}{\partial x^2}+\rho\nu x f(y)\frac{\partial^2 P}{\partial x\partial y}+\frac{\nu^2}{2}\frac{\partial^2 P}{\partial y^2}+$$
$$\left(x\frac{\partial P}{\partial x}-P\right)+\left[\varepsilon^{-1}(m-y)-\nu\Lambda(t,x,y)\right]\frac{\partial P}{\partial y}=0, \tag{1}$$
$$P(T,x,y)=h(x),$$

with $T=rT_E$, $\varepsilon=r\alpha^{-1}$ and $\nu=r^{-1/2}\beta$.

We observe that $\varepsilon=r/\alpha$ is the dimensionless inverse of the rate of mean-reversion and, hence, that small ε means *fast mean-reversion*. Moreover, $\nu=r^{-1/2}\beta$ is the dimensionless vol-vol. For the readers of (Fouque, Papanicolaou, & Sircar 2000) we remark the notation used there is somewhat different of ours. More specifically, they work with the vol-vol $\widetilde{\nu}=\nu\varepsilon^{1/2}$

If the volatility is fast mean-reverting, then the market will see, to leading order, an effective constant volatility plus small corrections. This is modeled by subsuming that the mean reversion time $\varepsilon := r/\alpha$ is small.

We assume in the ensuing analysis that: f is bounded away from zero and from above; Λ is independent of x; (this can actually be proved). We single out the following scale regimes for ν in (1):

1. $\nu^2\sim\varepsilon^{-1}$ corresponds to the scaling considered in FPS (Fouque, Papanicolaou, & Sircar 2000) and leads to a balance between the terms
$$\frac{\nu^2}{2}\frac{\partial^2 P}{\partial y^2} \text{ and } \varepsilon^{-1}(m-y)\frac{\partial P}{\partial y}\ .$$
2. $\nu^2\ll\varepsilon^{-1}$. See (Souza & Zubelli 2006).

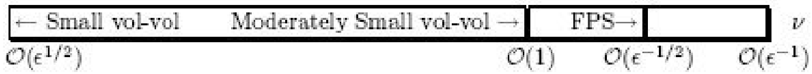
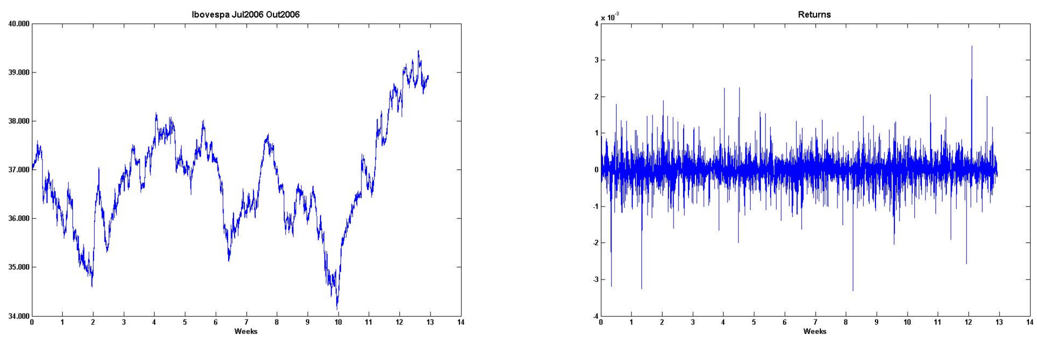


Figure 1: Asymptotic line with various distinguished regimes

3 Numerical Results from IBOVESPA

We performed a statistical analysis of the IBOVESPA index along the lines of that presented in (Fouque, Papanicolaou, & Sircar 2000). In what follows we report the results of such analysis. [1] Using the intra-day time series of the prices X_n we take 5 minutes averages $\widehat{X}_n$



(a) IBOVESPA Index. (b) IBOVESPA Returns

Figure 2: IBOVESPA for the Period July 2006 through October 2006.

and consider the asset price *returns* given by

$$D_n = \frac{2(\widehat{X}_n - \widehat{X}_{n-1})}{\sqrt{\Delta t}(\widehat{X}_n + \widehat{X}_{n-1})}$$

As in (Fouque, Papanicolaou, & Sircar 2000), we assume a expOU dynamics for the driving process. Thus, we analyzed the logarithm of the returns, passing through a ten point median filter. The main reason to do this filtering is to excluded outliers in the data. Finally, we compute the *variogram* $V_j^N = \frac{1}{N}\sum_{n=1}^{N}(L_{n+j} - L_n)^2$ where j is the lag and N is the total number of the points. It turns out that the variogram is an estimator to

$$2c^2 + 2\widetilde{\nu}_f^2(1 - \exp(-j\alpha\Delta t)) \tag{2}$$

where c is a constant and $\widetilde{\nu}_f$ is the estimated $\widetilde{\nu}$ for a given f.

We computed the mean reversion rate α and the parameter $\widetilde{\nu}_f^2 = \beta^2/(2\alpha) = (1/2)\nu^2\varepsilon$. Their numerical result is shown in the Table 1. These results imply a mean-reversion rate of approximately 1.3 days, which certainly indicates fast-mean reversion. However, the ratio between ν^2 and α is about 0.4. Thus, it is not a priori clear whether we are in the FPS vol-vol regime or in a moderately small vol-vol regime.

[1] We thank the computational work of Cassio A.M. Alves.

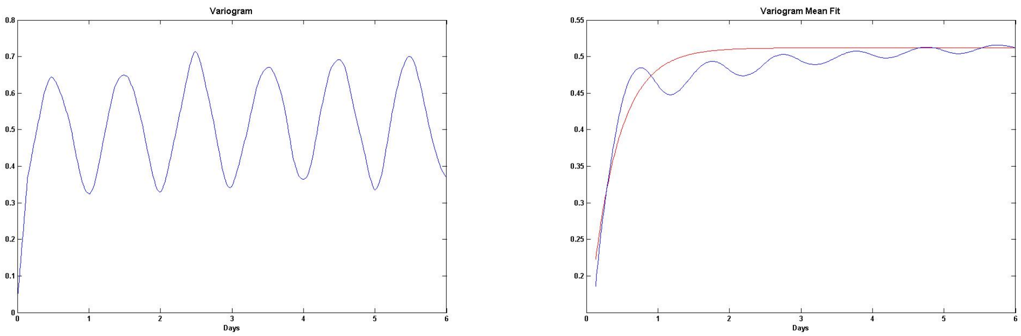


Figure 3: Variograms for the IBOVESPA

α	652.90
$\widetilde{\nu}_f^2$	0.231

Table 1: The estimated parameters for the IBOVESPA index during the period July 2006 through October 2006.

4 Small and Moderately Small Vol-Vol

As an example we consider the regime given by $\nu = O(1)$. We follow a multiple-scale approach and introduce the change $t \to T - \varepsilon s - \tau$. Under this hypothesis, our equation becomes

$$\varepsilon^{-1}\mathcal{H}_0 P^\varepsilon + (\mathcal{H}_1 + \mathcal{H}_2)P^\varepsilon = 0 \tag{3}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= -\frac{\partial}{\partial s} + (m-y)\frac{\partial}{\partial y},\\
\mathcal{H}_1 &= -\frac{\partial}{\partial \tau} + \frac{1}{2}(f(y))^2 x^2 \frac{\partial^2}{\partial x^2} + \left(x\frac{\partial}{\partial x} - \cdot\right),\\
\mathcal{H}_2 &= \nu_0 \rho x f(y)\frac{\partial^2}{\partial x \partial y} - \nu_0 \Lambda(t,x,y)\frac{\partial}{\partial y} + \frac{\nu_0^2}{2}\frac{\partial^2}{\partial y^2}.
\end{aligned}$$

Formally, we write

$$P^\varepsilon = P_0(s,\tau,x,y) + \varepsilon P_1(s,\tau,x,y) + \varepsilon^2 P_2(s,\tau,x,y) + O(\varepsilon^3). \tag{4}$$

One now proceeds substituting (4) on (3) and analyzing the terms of same order. For details, see (Souza & Zubelli 2006). We find the following expansion for the price

$$\begin{aligned}
P^\varepsilon(s,\tau,x,y) = P_0 + \varepsilon \Bigg\{ &\frac{\partial^2 P_0(\tau,x)}{\partial x^2}\int_y^{(y-m)\mathrm{e}^{-s}+m} \frac{\mathcal{H}_1^z P_0(\tau,x)}{m-z}\mathrm{d}z - \tau\Bigg[2\rho\nu_0 f^2(m) f'(m)\frac{x^3}{2}\frac{\partial^3 P_0}{\partial x^3}\\
&+\frac{x^2}{2}\frac{\partial^2 P_0}{\partial x^2}\Bigg(\frac{\nu_0^2}{4}\left(f'^2(m) + f(m)f''(m)\right)\\
&+4\rho\nu_0 f^2(m)f'(m) - \nu_0\Lambda(m)f(m)f'(m)\Bigg)\Bigg]\Bigg\},
\end{aligned} \tag{5}$$

whre P_0 satisfies a Black-Scholes with effective volatility $\bar{\sigma}$ given by

$$\bar{\sigma} = f(m).$$

Also, we get the following implied volatility I

$$I = \sigma + \frac{\varepsilon}{2\sigma(\tau+\varepsilon s)}\left\{\tau\left[A - B\frac{(1+3\sigma^2/2)}{\sigma^2} + A\frac{\log(x/K)}{\tau+\varepsilon s}\right] + +\varepsilon^{1/2}C\upsilon(1-\mathrm{e}^{-s})\right\} + O(\varepsilon^2), \tag{6}$$

$$\begin{aligned} A &= 2\rho\nu_0 f(m)f'(m), \\ B &= \frac{\nu_0^2}{4}\left(f'^2(m) + f(m)f''(m)\right) + 4\rho\nu_0 f^2(m)f'(m) - \nu_0\Lambda(m)f(m)f''(m), \\ C &= 2f(m)f'(m). \end{aligned}$$

Remarks: Notice that (6) is asymptotic both when $T-t$ is order one, and when $T-t = O(\varepsilon)$. In the former, we have $\tau \neq 0$. Thus (6) becomes

$$I = \sigma + \frac{\varepsilon}{2\sigma}\left(B - \frac{A(1+3\sigma^2/2)}{\sigma^2} + A\frac{\log(x/K)}{\tau}\right) + O(\varepsilon^{3/2}).$$

In the latter, we have $\tau = 0$. In this case, (6) becomes

$$I = \sigma + \varepsilon^{1/2}\frac{\upsilon}{2\sigma}C\frac{(1-\mathrm{e}^{-s})}{s} + O(\varepsilon).$$

We also analyzed the case of small vol-vol ($\nu = O(\varepsilon^{1/2})$) in (Souza & Zubelli 2006).

5 Conclusions

We obtained the asymptotic behavior of the solutions to the Equation (1) under different regimes. The regime $\nu^2 \sim \varepsilon^{-1}$ corresponds to the scaling considered in (Fouque, Papanicolaou, & Sircar 2000) and leads to a balance between the terms $\frac{\nu^2}{2}\frac{\partial^2 P}{\partial y^2}$ and $\varepsilon^{-1}(m-y)\frac{\partial P}{\partial y}$. The regime $\nu^2 \ll \varepsilon^{-1}$ is analyzed in further detail in our work (Souza & Zubelli 2006). We focused on the vol-vol of order $\varepsilon^{1/2}$ and a vol-vol of order ε. Identified two time scales: $O(1)$ and $O(\varepsilon^{-1})$. We found, in the latter regime, that the first order correction to the option prices depends on the state of the volatility process (y) and also on a fast time scale.

These corrections are not the small variance limits of those found in (Fouque, Papanicolaou, & Sircar 2000). The correction for either case shows that the fast scale part of the solution is governed by a transport equation. This can be interpreted as a terminal layer, which differs in spirit from the one found in (Howison, Rafailidis, & Rasmussen 2004) for the specific case of $\nu = O(\varepsilon^{-1/2})$, $\rho = 0$, and $\Lambda = 0$. Furthermore, the solution displays an outer and an inner layer, w.r.t. y. If f is constant outside a compact set of length asymptotically less than ε^{-1}, then the outer layer solution is y-independent.

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