Asymptotic Behavior of Stochastic Volatility Models

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Introduction and Outline: The smile curve and the bursty behavior of volatility is still a challenge and a source of interesting modeling problems in finances. The empirical remark that volatility tends to fluctuate at different levels and seems to mean-revert along a derivative contract life time led many authors to consider stochastic volatility market models [FPS00, Hes93, HW87, Wig87, SS91]. However, such stochastic volatility models introduce difficulties that cannot be analyzed satisfactorily unless one carefully takes into account the different time scales involved. This problem led [FPS00] to a very effective and practical way of correcting the computed prices in the Black-Scholes model so as to accomodate for the volatility under fast mean reversion.

In the present work, we explore a different asymptotic regime of the stochastic volatility model analyzed in [FPS00], discuss its implications and relevance.

The outline of the paper is the following: We start with some background material on stochastic volatility models and scaling so as to state some of the results in [FPS00]. Then, we briefly present a different scaling and describe our results.

Background on Stochastic Volatility Models: We start by brifely reviewing the classical Black-Scholes (B-S) market model so as to fix the notation. We denote by β a riskless asset (bond or insured bank deposit) and by X a risky asset. In the classical B-S model the assets undergo the following dynamics

$$\mathrm{d}\beta_t = r\beta_t \mathrm{d}t \qquad \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t$$

where W_t is the standard Brownian Motion. Let $P(t,x)$ denote the price of an European option at time t and current stock value x. Standard replication and non-arbitrage arguments lead to the classical Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + \left(r\frac{\partial P}{\partial x} - P \right) = 0 \qquad\qquad P(T,\cdot) = h \tag{1}$$

where h is the payoff at time T.   In [FPS00] the following dynamics for the risky asset is studied and motivated by the need of explaining a number of empirical observations

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \qquad \sigma_t = f(Y_t) \qquad dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t a$$

where $\widehat{Z}_t$ is a linear combination of two independent Brownian motions (W_t) and (Z_t). In this model, the risky asset's volatility is controlled by a stochastic process $y = Y_t$, which could be thought of as a hidden process. Such process Y_t, in turn, undergoes an Ornstein-Uhlenbeck dynamics. This choice is motivated by the empirical remark that the volatility tends to return to a historical level after some time. The return rate to such mean is denoted by α.

Let $P = P(t, x, y)$ be the price of an European option at time t given that the current stock price is x and its driving state is y. Once again, using a non-arbitrage argument it is argued in [FPS00] that $P(t, x, y)$ satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial x\partial y} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2} + r(x\frac{\partial P}{\partial x} - P) + \left(\alpha(m-y) - \beta\Lambda(t,x,y)\right)\frac{\partial P}{\partial y} = 0 \tag{2}$$

where

$$\Lambda(t, x, y) = \rho\frac{\mu - r}{f(y)} + \gamma(t, x, y)\sqrt{1-\rho^2}$$

with final condition $P(T, x, y) = h(x)$

Equation (2) can be interpreted considering the operator

$$\begin{aligned}&\frac{\partial}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) + \\ &+ \rho\beta x f(y)\frac{\partial^2}{\partial x\partial y} \\ &+ \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y} - \beta\Lambda\frac{\partial}{\partial y}\end{aligned}$$

The first line consists of the standard Black-Scholes operator with (stochastic) volatility $f(y)$. The second one consists of a correlation term. The third one is the generator for the O-U process added to a premium term associated to the market price of volatility risk.

Furthermore, we may regard γ as the risk premium factor from the second source of randomness which is driving the volatility (Z_t).

The Rescaled Equation: One key empirical remark in a large number of financial situations is the presence of multiple time scales. See for example [FPS00, FPSS03b, FPSS03a]. This is modeled by subsuming that the mean reversion time $\epsilon := 1/\alpha$ is small as compared to the other time scales. After introducing such scaling, Equation (2) becomes

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)\right)P^\epsilon = 0\ , \tag{3}$$

where

$$\mathcal{L}_0 =\nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \ ,$$
$$\mathcal{L}_1 =\rho\sqrt{2}xf(y)\frac{\partial^2}{\partial x \partial y} - s(y)\frac{\partial}{\partial y} \ ,$$
$$\mathcal{L}_2 =\frac{\partial}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \ ,$$

$\nu^2 := \beta^2/(2\alpha)$, and $s(t,x,y) := (\beta/\alpha)\Lambda(t,x,y)$. Furthermore, ν and $s(t,x,y)$ are assumed to be $\mathcal{O}(1)$. [FPS00] considered the following formal expansion

$$P^{\epsilon} = P_0 + \epsilon^{1/2}P_1 + \epsilon P_2 + \epsilon^{3/2}P_3 + \mathcal{O}(\epsilon^2)$$

After substituing such expansion into Equation (3) and grouping terms of same order they get

$$\mathcal{O}(\epsilon^{-1}) \qquad \mathcal{L}_0 P_0 = 0 \qquad \Rightarrow P_0 = P_0(t,x). \tag{4}$$
$$\mathcal{O}(\epsilon^{-1/2}) \qquad \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0 \qquad \Rightarrow P_1 = P_1(t,x). \tag{5}$$
$$\mathcal{O}(1) \qquad \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0 \qquad \Rightarrow \mathcal{L}_0 P_2 = -\mathcal{L}_2 P_0. \tag{6}$$

The $\mathcal{O}(1)$ equation implies the solvability condition $\langle \mathcal{L}_2 P_0 \rangle = 0$ upon applying the Fredholm alternative, where $\langle g \rangle := \int g(y)\Phi(y)\mathrm{d}y$ where $\mathcal{L}_0\Phi = 0$. Applying the solvability condition we get

$$\langle \mathcal{L}_2 \rangle = \frac{\partial P_0}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \left(r\frac{\partial P_0}{\partial x} - P_0\right) = 0$$

where $\sigma^2 = \langle f \rangle$ is an effective volatility. The next order leads to

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0 \ .$$

Once again, applying the solvability condition we get

$$\langle \mathcal{L}_2 P_1 \rangle = \tilde{V}_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \tilde{V}_3 x^3 \frac{\partial^3 P_0}{\partial x^3}$$

and

$$P_1 = -(T-t)\left[\tilde{V}_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + \tilde{V}_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right].$$

Thus, the explicit formula for the corrected price is given by

$$P = P_0 - (T-t)\left[V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right] + \mathcal{O}(\epsilon),$$

where $V_2 = \epsilon\tilde{V}_2$ and $V_3 = \epsilon\tilde{V}_3$.

The Problem under Consideration: In the present contribution, we consider the question: what happens if, differently from [FPS00], we assume that $\nu = \mathcal{O}(\epsilon)$?

This is a relevant question because such coefficient ν^2 represents the volatility of the volatility (vol-vol) and pressumably in some markets this might be a more realistic scenario than the underlying assumption made in [FPS00] that ν is of order 1.

Under the above hypothesis $\nu = \mathcal{O}(\epsilon)$, Equation (3) becomes

$$\epsilon^{-1}\mathcal{L}_0 P^\epsilon + \mathcal{L}_1 P^\epsilon + \epsilon^{1/2}\mathcal{L}_2 P^\epsilon + \epsilon \frac{\partial^2 P^\epsilon}{\partial y^2} = 0$$

where now

$$\begin{aligned}
\mathcal{L}_0 =& (m-y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1 =& \frac{\partial}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), \\
\mathcal{L}_2 =& \rho\sqrt{2}x f(y)\frac{\partial^2}{\partial x \partial y} - s(y)\frac{\partial}{\partial y}.
\end{aligned}$$

Our perturbation analysis yields

$$P^\epsilon = P_0(t,x,y) + \epsilon^{1/2}P_1(t,x,y) + \epsilon P_2(t,x,y) + \epsilon^{3/2}P_3(t,x,y)\mathcal{O}(\epsilon^2) \tag{7}$$

At level $\mathcal{O}(\epsilon^{-1})$, we get $\mathcal{L}_0 P_0 = 0$, which implies $P_0 = P_0(t,x)$. At level $\mathcal{O}(\epsilon^{-1/2})$ we have that $\mathcal{L}_0 P_1 = 0$, and hence that $P_1 = P_1(t,x)$. For $\mathcal{O}(\epsilon^0)$, we have $\mathcal{L}_0 P_2 + \mathcal{L}_1 P_0 = 0$. Here, the solvability conditions yields $\mathcal{L}_1^m P_1 = 0$, $\mathcal{L}_1^m$ is B-S operator with $\sigma = f(m)$, and terminal condition $P_0(T,x) = h(x)$. The solvability condition for P_3 plus a final condition on P_1 implies $P_1(t,x,y) \equiv 0$. Finally, at order $\mathcal{O}(\epsilon)$, the solvability condition for $\mathcal{L}_0 P_4 = -\mathcal{L}_1 P_2 - \mathcal{L}_2 P_1 - \frac{\partial^2 P_0}{\partial y^2}$, simplifies to $\mathcal{L}_1^m P_2 = 0$ and $P_2(T,x,y) = 0$. We remark that in the present context, the solvability condition cannot be satisfied. In this case one needs to consider a *Terminal Layer*.

Conclusions: In [FPS00] a far-reaching asymptotic analysis of stochastic volatility models was developed under a number of hypothesis, including that the vol-vol coefficient is of the same order ($\mathcal{O}(1)$) of the mean reversion time.

In the present work we show that there exists a distinguished asymptotic limit of the stochastic volatility model different from that studied in [FPS00] provided one assumes that the vol-vol coefficient ν^2 is small as compared to the mean reversion time of the volatility. This result shows the possibility of exploring more complex situations than those studied in [FPS00]. In particular, we find that there exists a *terminal layer* in the asymptotic regime of the price correction of order $\epsilon = 1/\alpha$. Furthermore, such correction is non-diffusive. One plausible interpretation of this would be that, in the regime under consideration and close to expiration time, the option price correction, P_1 in (7), would not be influenced by the volatility.

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