Real Option Pricing with Mean-Reverting Investment and Project Value

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Abstract

In this work we are concerned with real option prices when the project value V_t and the investment value I_t undergo a mean-reverting stochastic dynamics. We consider the question of finding the dynamics for which an investment trigger curve, based on the ratio V_t/I_t, can be determined.

For a particular class of mean-reverting processes, we show that the investment frontier can be represented by such a ratio. In particular, the dynamics of the ratio is also mean-reverting.

For more general dynamics, which might include jumps, the above reductions do not seem to be possible, and a Fast Fourier Stepping Method, developed by Jackson, Jaimungal, and Surkov (2008) and Jaimungal and Surkov (2009), is discussed instead.

Key-words: Real Options; Mean-Reverting; Stochastic Investment

1 Introduction

Quantitative methods to analyze the option to invest in a project enjoy a long and distinguished history. The classical work of McDonald and Siegel (1986) (see also Dixit and Pindyck (1994)) investigates the problem from the point of view of derivative pricing and assigns the value of the option to invest as

$$\text{value} = e^{-rT}\mathbb{E}\left[(V_T - I_T)_+\right] \ . \tag{1.1}$$

Here, the expected value is taken under an appropriate risk-adjusted measure. Furthermore, V_T and I_T represent the project's value and the amount to be invested, respectively, at time T.

If the project can be started at anytime, then (1.1) is modified to its American counterpart. In this case, the maturity date T is replaced by a stopping time τ ($0 \le \tau \le T$) and the investor chooses the stopping time to maximize the option's value. As such, the problem becomes a free

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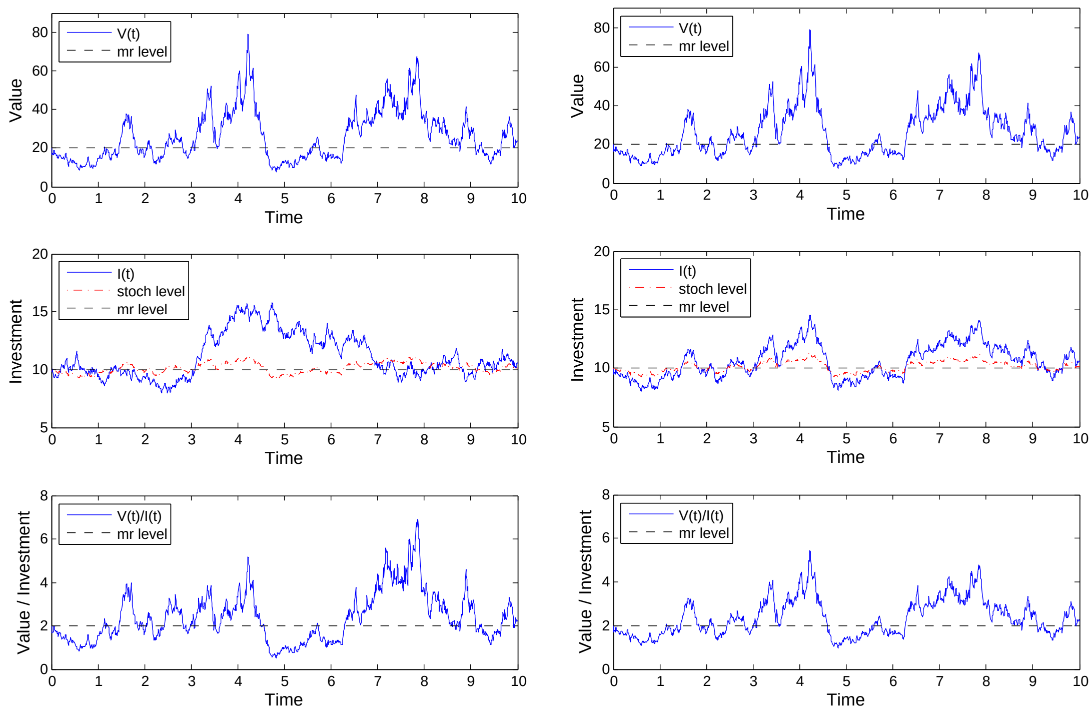
boundary problem in which the optimal strategy is computed simultaneously with the option's value.

Traditionally, the project value is assumed to be a geometric Brownian motion (GBM) and the investment amount is constant or deterministic, as in the pioneering work of Tourinho (1979). Stochastic investment amounts have also been investigated previously: the case of a GBM driven investment, when the opportunity to invest does not expire in time (i.e. a perpetual option), is treated in McDonald and Siegel (1986) (see also Berk, Green, and Naik (1999)). More recently, Elliott, Miao, and Yu (2007) have investigated the case of regime switching investment costs for the option in perpetuity. Perpetuities have also been investigated with a mean-revearting CIR model as the project value and constant investment by (Ewald and Wang 2007). It should be also pointed out that the problems that arise with uncertain investment are similar to those found in exchange options, as in Margrabe (1978), and in uncertain payoffs, as in Fischer (1978).

Much of these works – e.g. McDonald and Siegel (1986) and Blenman and Clark (2005) – assume that the amount to be invested is a GBM. The latter may be a good model for the project value in certain circumstances, since in many cases it represents a net present value. On the other hand, as already noticed in McDonald and Siegel (1986), the investment costs are typically prices of commodities, and thus are expected to revert to an equilibrium level. Furthermore, in situations where the cashflows of the project are directly linked to commodities, the project value is also expected to approach an equilibrium level. One such situation is the valuation of the option to the invest in an oil field. Like most commodities, oil prices tend to mean-revert, and as a direct result the value of investment in an oil field is also mean-reverting. Consequently, it would not be appropriate to use GBM to model the value of such a project. Of course, several authors have noticed this and mean-reverting processes have been considered, such as Metcalf and Hasset (1995) and Sarkar (2003). However, combining mean-reverting project value with mean-reverting investment amount has not been considered up to now. There are good reasons for the amount to be invested to be mean-reverting. Consider an oil company which is contemplating to invest in a recently found oil field. The oil company's profits and therefore the amount available to invest, will tend to mean-revert.

2 Trigger curves for mean-reversion investments

The difficulty with allowing both project value V_t and investment amount I_t to mean-revert lies in the fact that the problem becomes two-dimensional and the optimal policy will depend on both V_t and I_t. In the case when both processes are GBM, the optimal policy depends only on the ratio V_t/I_t and the value of the option becomes homogeneous in I_t (or V_t) — this was observed in McDonald and Siegel (1986) and it seems that this *trigger procedure* has become a paradigm in Real Options pricing. See Dixit and Pindyck (1994) for a review of these triggers for perpetual options with both GBM and mean-reverting project values but constant investment. We therefore seek a new mean-reverting model which produces the qualitative features of mean-reverting V_t and I_t while maintaining the homogeneity of the solution. To this end, our model assumes the



(a) $\rho = 0$ (b) $\rho = 1.0$

Figure 2.1: Two sample paths with differing levels of correlation generated by the same uncorrelated Brownian source. The lines label *mr level* are the long-run mean-reverting levels for the value and investment, while *stoch level* is the instantaneous mean reversion level of the investment. The model parameters are: $\alpha = 1$; $\theta = \ln(20)$; $\sigma_X = 0.8$; $\beta = 1.1$; $\phi = \ln(10)$;$\sigma_Y = 0.2$; and $\rho = 0.5$.

following

$$V_t = e^{\theta + X_t}, \tag{2.1a}$$

$$dX_t = -\alpha X_t \, dt + \sigma_X \, dW_t^X, \tag{2.1b}$$

$$I_t = e^{\phi + Y_t}, \tag{2.1c}$$

$$dY_t = -((\alpha - \beta) X_t + \beta Y_t) \, dt + \sigma_Y \, dW_t^Y, \tag{2.1d}$$

Here, W_t^X and W_t^Y are, in general, correlated Brownian motions with correlation ρ. As usual we work on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{(\mathscr{F}_t)_{0 \le t \le T}\}$ ($\mathscr{F}_t = \sigma((W_s^X, W_s^Y)_{0 \le s \le t})$) is the natural filtration generated by the driving Brownian motions and $\mathbb{P}$ is the statistical (historical) probability measure.

In this model, the value V_t of the project mean-reverts to a long-run level θ, while the investment I_t available for the project instantaneously mean-reverts to a stochastic level $\eta_t := \exp\{\phi - \frac{\alpha - \beta}{\beta} X_t\}$. However, the process X_t itself mean-reverts to zero, implying that $\exp\{\phi\}$ is the true long-run level of the investment process. This coupling of investment and value is not entirely artificial. In fact, it is quite reasonable to assume that the amount available for the investment is tied in some way to the value of the project itself. Nonetheless, this coupling of investment can be minimized by appropriate choices of the model parameters.

In Figure 2.1, two sample paths for the value and investment are presented. The sample paths were both generated from the the same two uncorrelated Brownian sample paths to highlight the effect of correlation. Panel (a) contains no correlation between the increments in the investment level and value; however, since the investment is instantaneously pulled to the stochastic level η_t, there is some feedback effect. In fact, the processes X_t and Y_t are cointegrated. Panel (b) illustrates the behavior when there is correlation one. Here it is clear that in addition to the attraction level, the pathwise behavior of the project's value and investment amount are strongly coupled. Having the flexibility to incorporate both features is quite desirable. Furthermore, note that the ratio of project value to investment for both correlations have very similar same path behaviour; however, the amplitude of the fluctuations are reduced in the correlated case.

Under the modeling assumption (2.1), the ratio V_t/I_t of the project's value and the amount invested is also a mean-reverting processes *and* the dynamics of this ratio depends only on the ratio itself. Specifically, notice that $\frac{V_t}{I_t} = e^{(\theta-\phi)+(X_t-Y_t)}$ and define $Z_t = X_t - Y_t$, then

$$dZ_t = -\beta\, Z_t\, dt + \sigma_X\, dW_t^X - \sigma_Y\, dW_t^Y \; . \tag{2.2}$$

This implies that the ratio can be modeled directly as a mean-reverting process with mean-reversion rate β and effective instantaneous variance of $\sigma^2 := \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$.

Note that, in order to focus the study on a mean-revearting value-to-investment ratio, we have not considered dividends associated with project. These issues can easily be addressed and do not significantly affect the framework; however we deligate such explorations to future work. Most interestingly, the mean-reversion nature of the investment already enjoys an early-exercise premium for the option, contrary to the constant case or pure geometric brownian case.

We now investigate the option to invest under the modeling framework (2.1). The Bermudan option to invest, where investment can only be exercised at discrete times $\{t_0, t_1, \dots, t_n\}$ (e.g. quarterly, monthly, or weekly), can priced recursively on the exercise dates as follows:

$$\begin{cases} p_{t_n}(V_{t_n}, I_{t_n}) = (V_{t_n} - I_{t_n})_+ \\ \qquad\qquad p_{t_{m-1}} = \max\left\{ e^{-r\Delta t_m}\mathbb{E}\left[p_{t_m}(V_{t_m}, I_{t_m}) \,|\mathscr{F}_{t_{m-1}}\right] ;\; (V_{t_{m-1}} - I_{t_{m-1}})_+ \right\}, \end{cases} \tag{2.3}$$

for $m = \{1, 2, \dots, n\}$. Let us proceed to describe how to value the option. First, we require

$$f_{t_{m-1}} \triangleq \mathbb{E}\left[(V_{t_m} - I_{t_m})_+ \,|\mathscr{F}_{t_{m-1}} \right] \tag{2.4}$$

$$= \mathbb{E}\left[(V_{t_m}/I_{t_m} - 1)_+ I_{t_m} \,|\mathscr{F}_{t_{m-1}} \right] \; . \tag{2.5}$$

If I_t were a geometric Brownian motion, then it would be straightforward to absorb I_t into a simple measure change – akin to a numeraire change. However, due to the mean-reverting behavior of I_t a more clever measure change is necessary to absorb it. To this end, introduce a new measure $\mathbb{P}^T$ via the Radon-Nikodym derivative process

$$\eta_t^T \triangleq \left(\frac{d\mathbb{P}^T}{d\mathbb{P}} \right)_t = \frac{\mathbb{E}[I_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_0]} \; . \tag{2.6}$$

Notice that $\eta_T^T = I_T/\mathbb{E}[I_T|\mathscr{F}_0]$ so that,

$$f_{t_{m-1}} = \mathbb{E}[I_{t_m}|\mathscr{F}_{t_{m-1}}]\; \mathbb{E}^{t_m}\left[(V_{t_m}/I_{t_m} - 1)_+ \,|\mathscr{F}_{t_{m-1}} \right] \tag{2.7}$$

$$= \mathbb{E}[I_{t_m}|\mathscr{F}_{t_{m-1}}]\; \mathbb{E}^{t_m}\left[(\xi_{t_m}^{t_m} - 1)_+ \,|\mathscr{F}_{t_{m-1}} \right] \tag{2.8}$$

where $\mathbb{E}^{t_m}[.]$ denotes expectation under the new measure $\mathbb{P}^{t_m}$ and

$$\xi_t^T \triangleq \frac{\mathbb{E}[V_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_t]} \; . \tag{2.9}$$

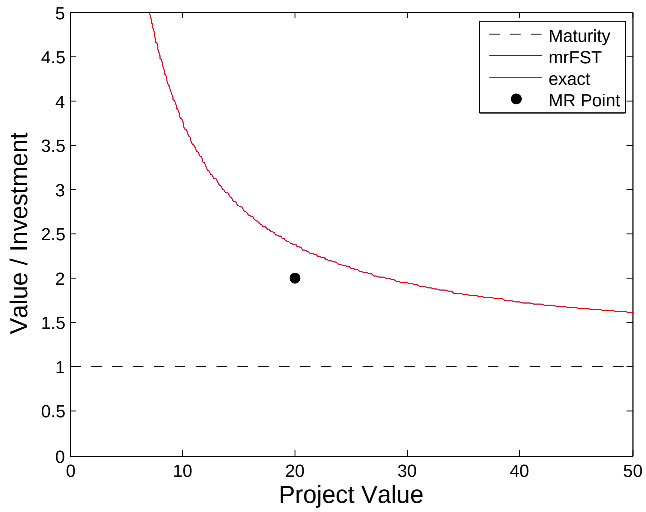


Figure 2.2: The optimal exercise boundary for 1 year remaining to maturity computed from the analytical expression in (2.11) and the mrFST method outlined in Section 3. The exercise region lies above the dashed line at maturity, and above the blue/red line at one-year prior to maturity. The model parameters are as in Figure 2.1.

In the appendix we demonstrate the ξ_t^T is a $\mathbb{P}^T$-martingale and in particular,

$$\frac{d\xi_t}{\xi_t} = \sigma_X e^{-\beta(T-t)} dW_t^{T,X} - \sigma_Y e^{-\beta(T-t)} dW_t^{T,Y} \tag{2.10}$$

where $W_t^{T,X}$ and $W_t^{T,Y}$ are correlated standard $\mathbb{P}^T$-Brownian motions. In particular, by applying the result in the appendix, the value of the option to invest after the first iteration is

$$f_{t_{n-1}} = \mathbb{E}[V_{t_n}|\mathscr{F}_{t_{n-1}}]\Phi\left(d_+(t_{n-1},t_n)\right) - \mathbb{E}[I_{t_n}|\mathscr{F}_{t_{n-1}}]\Phi\left(d_-(t_{n-1},t_n)\right) \; . \tag{2.11}$$

Expressions for the remaining expectations and $d_\pm$ can be found in equations (A.10)-(A.11) and (A.16), and $\Phi(\cdot)$ denotes the standard normal cdf.

Through recursive application of the described measure change, the option can be evaluated through a series of one-dimensional problems because the ratio V_{t_n}/I_{t_n} depends solely on the Z_t process and not X_t and Y_t individually. More specifically, a one-dimensional binomial tree can be developed for the ratio process; however, a new measure must be used between each exercise date. This does not pose any real problems and we are able to compute the optimal exercise policy as a function of V_t/I_t and I_t.

In Figure 2.2, we show a plot of optimal exercise boundary assuming exercise can occur at maturity and one-year prior to maturity only. The results are also compared with the mrFST method outlined in the next section. At maturity, exercise occurs whenever $V > I$ (the dashed line), while at one-year prior to maturity the exercise region lies above the red/blue lines. The mean-reversion point is also shown in the diagram to illustrate that this option is very often in-the-money.

3 Beyond mean-reversion

The above procedure is appropriate when the process does not contain any jumps; however, if jumps are present, then alternative methods must be used. Firstly, jumps render a tree approximation inadequate – multinomial trees are possible, but inaccuracies arise quickly. Furthermore, finite-difference schemes require inverting dense matrices resulting in large slowdowns and potential errors due to truncation of large jumps. Secondly, the measure changed induced by a jump process is more complicated, and although it is possible to derive the appropriate change, tractability is lost. Instead, we will now describe a variant of the mean-reverting Fourier Space Time-Stepping method of Jaimungal and Surkov (2009) appropriate for this real-options context and which is also easily extensible to incorporate jumps. See also Jackson, Jaimungal, and Surkov (2008) for the FST method without mean-reversion.

Consider the value of the option to invest in between two decision dates, i.e. $t \in (t_{m-1}, t_m)$, without the discount value:

$$\overline{p}_t = \mathbb{E}\left[p_{t_m}(X_{t_m}, Y_{t_m})\,|\mathscr{F}_t\right] \; . \tag{3.1}$$

Notice that, without loss of generality, we have chosen to write the option value in terms of the "log" processes X_t and Y_t. When viewed as a process $\overline{p}_t$ is a $\mathbb{P}$-martingale, consequently it satisfies the PDE

$$\begin{cases} (\partial_t + \mathcal{L})\,\overline{p}(t, X, Y) = 0 \\ \qquad \overline{p}(t_m, X, Y) = p_{t_m}(X, Y) \end{cases} \tag{3.2}$$

Here, $p_{t_m}(X, Y)$ is already known from the previous step in the iteration and $\mathcal{L}$ is the infinitesimal generator of the process (X_t, Y_t)

$$\mathcal{L} = -\,\alpha\partial_X + \tfrac{1}{2}\sigma_X^2\partial_{XX} - ((\alpha-\beta)X + \beta Y)\partial_Y + \tfrac{1}{2}\sigma_Y^2\partial_{YY} + \rho\sigma_X\sigma_Y\partial_{XY} \; . \tag{3.3}$$

By introducing the $2D$-Fourier transform of $\overline{p}(t, X, Y)$ with respect to the X and Y variables, the PDE can be solved explicitly (see Jaimungal and Surkov (2009)) resulting in

$$\overline{p}(t, X, Y) = \mathcal{F}^{-1}\left[\mathcal{F}\left[\widetilde{p}(t_m, X, Y)\right](\omega_1, \omega_2)e^{\Psi((t_m - t), \omega_1, \omega_2)}\right] \tag{3.4}$$

Here, $\widetilde{p}(t_m, X, Y) = p_{t_m}(Xe^{-\alpha(t_m-t)}, Xe^{-\alpha(t_m-t)} + (Y - X)e^{-\beta(t_m-t)})$, Ψ is related to the characteristic function of the generating process and in this case is specifically

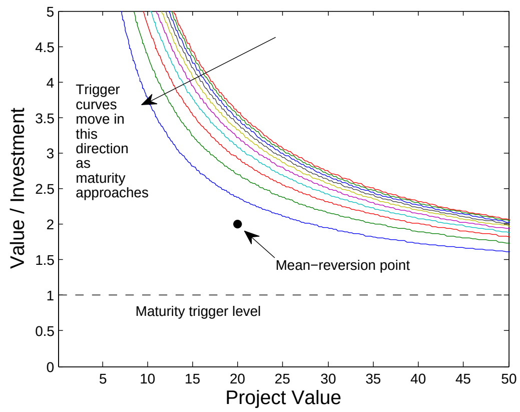
$$\begin{aligned} \Psi(s, \omega_1, \omega_2) = &-\tfrac{1}{2}\sigma_X^2\left(\tfrac{e^{2\alpha s}-1}{2\alpha}(\omega_1+\omega_2)^2 + \tfrac{e^{2\beta s}-1}{2\beta}\omega_2^2 - 2\tfrac{e^{(\alpha+\beta)s}-1}{\alpha+\beta}\omega_2(\omega_1+\omega_2)\right) \\ &-\tfrac{1}{2}\sigma_Y^2\tfrac{e^{2\beta s}-1}{2\beta}\omega_2^2 \\ &-\rho\sigma_X\sigma_Y\left(\tfrac{e^{(\alpha+\beta)s}-1}{\alpha+\beta}\omega_1 - \tfrac{e^{2\beta s}-1}{2\beta}\omega_2\right)\omega_2 \end{aligned} \tag{3.5}$$

and $\mathcal{F}[.]$ and $\mathcal{F}^{-1}[.]$ represent Fourier and inverse Fourier transforms respectively.

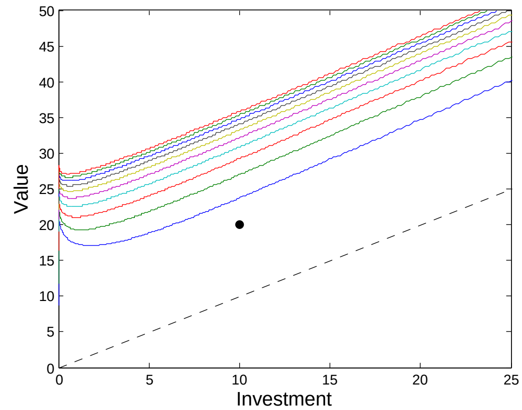
Through the above representation, a recursive formulation for the value at any given time step can be written as

$$p_{t_{m-1}}(X, Y) = \max\left\{e^{-r\Delta t_m}\mathcal{F}^{-1}\left[\mathcal{F}\left[\widetilde{p}(t_m, X, Y)\right](\omega_1, \omega_2)e^{\Psi(\Delta t_m, \omega_1, \omega_2)}\right]\;;\;(V_{t_{m-1}} - I_{t_{m-1}})_+\right\} \tag{3.6}$$

By comparison with the intrinsic value, the optimal strategy can be computed numerically through two fast Fourier transforms which approximately evaluate the Fourier and inverse transforms. This



(a) V/I versus V



(b) V versus I

Figure 3.3: The trigger boundaries from maturity to year ten with yearly decisions. Model parameters are as in Figure 2.2.

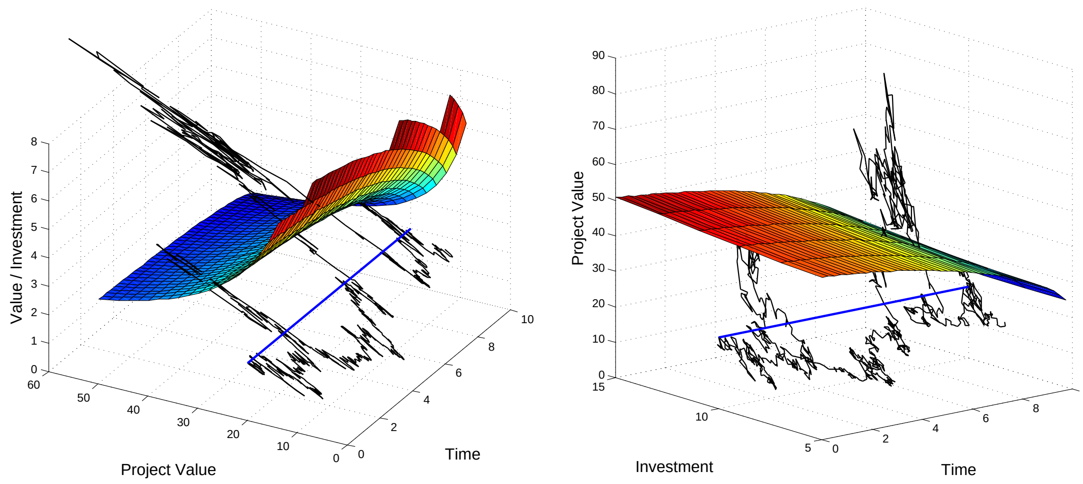
procedure is far more efficient than a tree or finite-difference scheme as it requires $O(N \log N)$ computations per exercise date, while finite difference schemes will require $O(MN)$ where M is the number of steps required between exercise dates. Furthermore, it is straightforward to incorporate jumps into the above representation – it will require a simple modification of the function Ψ – while tree or finite-difference methods will run into stability and computational issues.

In Figure 3.3, we plot the sequence of trigger curves for a ten year option to invest assuming investment can be made only once a year. Panel (a) illustrates the trigger curves as a function of V/I versus I while panel (b) provides a view of the curves V as a function of I. As maturity approaches, the trigger curves move toward the exercise trigger of $V/I = 1$, however, due to the mean-reversion point lying well within the exercise trigger region, the early trigger curves lie significantly above the line $V/I = 1$.

In Figure 3.4, we plot the trigger surface for a ten year option to invest assuming investment can occur daily. The solid blue line indicates the mean-reversion level, while the black random path is a sample of the process. Interestingly, when viewed as in panel (a), the sample paths appear to move mostly in a plane almost perpendicular to the trigger curve. When the sample path crosses the surface, investment in the project should occur. From the panel (a) viewpoint it is clear that investment should have occurred near year 2, while from the panel (b) viewpoint, this threshold crossing is not so evident. This suggests that using the V/I versus V perspective is advantageous.

4 Conclusions

In this work, we have addressed the problem of the decision of investing, when both the value of the project and the investment follows a mean-reverting dynamics. In this case, the optimal policy depends on both the value of the project and the investment level, rather than just on their ratio. The former is known to be the case when the value and the investment follow GBM dynamics. This phenomenon precludes the use of a trigger curve for determining the investment



(a) V/I versus V (b) V versus I

Figure 3.4: The trigger surface together with a sample path for a ten year option to invest with daily exercise decisions. Model parameters are as in Figure 2.1.

frontier, which has been recognized, since the work by McDonald and Siegel (1986), as a specially convenient representation. For a particular class of mean-reverting dynamics, we are able to show that such an investment frontier can be represented just by the ratio between the project value and the investment level. In particular, the dynamics of the ratio is also mean-reverting. For more general dynamics, which might include jumps, such reductions do not seem to be possible. Nonetheless, the Fourier Space Time-Stepping method, developed by Jackson, Jaimungal, and Surkov (2008) and Jaimungal and Surkov (2009), can be used to numerically explore the trigger levels in such models.

Acknowledgments

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A Option Pricing Formulae

In this appendix we derive the value of the option to invest in a project with stochastic investment and project value. The value is

$$Opt_t = e^{-r(T-t)}\mathbb{E}[(V_t - I_t)_+ \,|\mathscr{F}_t] = e^{-r(T-t)}\mathbb{E}^T[\left(\frac{V_t}{I_t} - 1\right)_+ |\mathscr{F}_t]\,\mathbb{E}[I_T|\mathscr{F}_t] \tag{A.1}$$

$$= e^{-r(T-t)}\mathbb{E}^T[(\xi_t - 1)_+ \,|\mathscr{F}_t]\,\mathbb{E}[I_T|\mathscr{F}_t] \tag{A.2}$$

where,

$$\xi_t \triangleq \frac{\mathbb{E}[V_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_t]} \,. \tag{A.3}$$

Note that ξ_t is a $\mathbb{P}^T$-martingale under any modeling assumptions for V_t and I_t (as long as I_t is strictly positive). This can be seen from the following computation ($0 \le s \le t$):

$$\begin{aligned}
\mathbb{E}^T[\xi_t|\mathscr{F}_s] &= \mathbb{E}\left[\frac{\mathbb{E}[V_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_t]} \cdot \frac{\mathbb{E}_t[I_T]}{\mathbb{E}_0[I_T]} \middle| \mathscr{F}_s \right] \Big/ \frac{\mathbb{E}[I_T|\mathscr{F}_s]}{\mathbb{E}[I_T|\mathscr{F}_0]} && \text{(A.4)}\\
&= \frac{\mathbb{E}\left[\mathbb{E}[V_T|\mathscr{F}_t]|\mathscr{F}_s\right]}{\mathbb{E}[I_T|\mathscr{F}_s]} && \text{(A.5)}\\
&= \frac{\mathbb{E}[V_T|\mathscr{F}_s]}{\mathbb{E}[I_T|\mathscr{F}_s]} && \text{(A.6)}\\
&= \xi_s \,. && \text{(A.7)}
\end{aligned}$$

For the model (2.1), we have

$$X_T = e^{-\alpha(T-t)}\, X_t + \sigma_X \int_t^T e^{-\alpha(T-u)}\, dW_u^X \tag{A.8}$$

$$\begin{aligned}
Y_T =& e^{-\beta(T-t)}\, Y_t + \left(e^{-\alpha(T-t)} - e^{-\beta(T-t)}\right) X_t \\
&+ \sigma_X \int_t^T \left(e^{-\alpha(T-u)} - e^{-\beta(T-u)}\right) dW_u^X + \sigma_Y \int_t^T e^{-\beta(T-u)}\, dW_u^Y
\end{aligned} \tag{A.9}$$

so that,

$$\mathbb{E}_t[V_T] = \exp\left\{\theta + e^{-\alpha(T-t)}\, X_t + \tfrac{\sigma_X^2}{4\alpha}\left(1 - e^{-2\alpha(T-t)}\right)\right\} , \tag{A.10}$$

$$\begin{aligned}
\mathbb{E}_t[I_T] = \exp\Big\{&\phi + e^{-\beta(T-t)}\, Y_t + \left(e^{-\alpha(T-t)} - e^{-\beta(T-t)}\right) X_t \\
&+\tfrac{1}{2}\sigma_X^2 \left[\tfrac{1-e^{-2\alpha(T-t)}}{2\alpha} + \tfrac{1-e^{-2\beta(T-t)}}{2\beta} - 2\tfrac{1-e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta}\right] \\
&+\tfrac{1}{2}\sigma_Y^2 \left[\tfrac{1-e^{-2\beta(T-t)}}{2\beta}\right] \\
&+\rho\sigma_X\sigma_Y \left[\tfrac{1-e^{-(\alpha+\beta)(T-t)}}{\alpha+\beta} - \tfrac{1-e^{-2\beta(T-t)}}{2\beta}\right]\Big\} \,.
\end{aligned} \tag{A.11}$$

Using Ito's lemma, and the fact that ξ_t is a $\mathbb{P}^T$-martingale, implies

$$\frac{d\xi_t}{\xi_t} = \sigma_X e^{-\beta(T-t)} dW_t^{T,X} - \sigma_Y e^{-\beta(T-t)} dW_t^{T,Y} \tag{A.12}$$

where $W_t^{T,X}$ and $W_t^{T,Y}$ are correlated standard $\mathbb{P}^T$-Brownian motions. As such,

$$\xi_T \stackrel{d}{=} \xi_t \exp\left\{-\tfrac{1}{2}\widetilde{\sigma}^2(t,T) + \widetilde{\sigma}(t,T) Z\right\} \tag{A.13}$$

where Z is a standard normal random variable and

$$\widetilde{\sigma}^2(t,T) \triangleq \left(\sigma_X^2 - 2\rho\sigma_X\sigma_Y + \sigma_Y^2\right) \tfrac{1-e^{-2\beta(T-t)}}{2\beta} \,. \tag{A.14}$$

Consequently, the value of the option to invest is

$$Opt_t = e^{-r(T-t)} \left\{\mathbb{E}_t[V_T]\Phi(d_+(t,T)) - \mathbb{E}_t[I_T]\Phi(d_-(t,T))\right\} , \tag{A.15}$$

with

$$d_\pm = \frac{\ln\left(\mathbb{E}_t[V_T]/\mathbb{E}_t[I_T]\right) \pm \frac{1}{2}\widetilde{\sigma}^2(t,T)}{\widetilde{\sigma}(t,T)} \,. \tag{A.16}$$

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