REAL OPTIONS UNDER FAST MEAN REVERSION STOCHASTIC VOLATILITY

MAX O. SOUZA AND JORGE P. ZUBELLI

Abstract. In this paper, we study the McDonald-Siegel (MS) model for real options under the assumption that the spanning asset undergoes a stochastic volatility dynamics that reverts to a historical value according to an Ornstein-Uhlenbeck process driven by a second source of uncertainty. In this case, the market is not complete, and valuation, even for a perfectly correlated asset, is not as straightforward as in the MS model. Nevertheless, it is possible to derive a pricing equation by risk-neutral arguments that depends on the so-called market risk premium. Under the further assumption that the driving volatility process is fast-mean reverting, we derive an asymptotic approximation for the value of a real-option. In such case, the model becomes very parsimonious and can be calibrated to real data.

1. Introduction

The use of mathematical tools from quantitative finance to evaluate corporate investments under uncertainty leads to a number of challenging mathematical problems that are of direct practical interest. In this article we pose and analyze one of such problems, namely a free-boundary partial differential equation associated to the option to delay an investment decision and to undertake it in the future [Mye77]. Such *real option* to defer investment has a value that can be, in turn, modeled and quantified by option theory in the context of derivatives of *American type*. Such *option to defer* is associated, for example, to problems where management holds a lease on valuable land or resources and it can wait until output prices justify investment. See [MS86, PSS88, Tou79, Tit85, IR92]. For the real option approach we refer to [MS86, Dix89, TM87, Pin91].

In this article we focus on the classical McDonald-Siegel approach under the condition of stochastic volatility and fast mean reversion of the spanning asset. The study of stochastic volatility models has been the subject of intense research throughout the last decades. See [HW87, Sco87, Wig87, SS91, HPS92, AN93, AM94, RB94, Hes93, Dua95, RPT00, RT96, MCC98, Tou99, ZA98, FPS00] and references therein.

The importance of stochastic volatility models can be explained by the fact that the classical Black-Scholes model suffers from a number of draw-backs: The log-normality of the underlying asset prices is not verified by statistical tests and the corresponding option prices are subject to the so-called smile-effect [Dup94]. On the other hand, in the case of stochastic volatility models, one can find heavier tails than those in the log-normal model of underlying prices, the distribution could be asymmetric and the smile curve appears naturally [RT96]. However, being more realistic comes with a price. Indeed, stochastic volatility models are much harder to analyze and could be impractical due to the appearance of the so-called market price of volatility risk.

Date: February 19th, 2008.

One effective approach for overcoming such difficulties and performing a practical analysis is to exploit the different scales involved, thus drawing from asymptotic techniques. This program has been implemented in a ground breaking way by [FPS00]. See also [FPS01b, FPS01a, FPSS03a, CFPS04, SZ07a].

We present the asymptotic analysis of the real option to defer investment for the McDonald-Siegel model under a fast mean reverting regime of stochastic volatility. We obtain following [FPS00] the first order correction of the price and exemplify how this price could be used by means of numerical simulations. The latter indicate that the presence of stochastic volatility tends to add value to the option to defer and to increase the optimal time to start investment.

The plan for this article is the following: In Section 2 we briefly review the key aspects of the McDonald-Siegel model. In Section 3 we present the asymptotic expansion of the stochastic volatility model under fast mean reversion. In Section 5 we present our numerical results, draw some conclusions and describe some directions for further research.

2. The McDonald-Siegel Model

In the present section we briefly review the McDonald-Siegel Model[MS86] for investments under uncertainty.

Suppose that a corporation is considering whether to launch a new project and let us assume that the estimated value of such project at a given time t is V_t. Suppose, furthermore that V_t evolves according to the stochastic differential equation

$$dV_t = \alpha V_t dt + \sigma_t V_t dW_t \ ,$$

where here, differently from the traditional MS model, we take σ_t to be a stochastic process driven by a (hidden) stochastic process Y_t. The process Y_t, on the other hand, evolves according to a dynamics of the form $dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$, where Z_t is a Brownian possibly correlated to W_t.

Assume that the fixed cost of launching such project are known and given by I. Two fundamental questions appear:

- How much is such opportunity worth?
- What is the optimal time to launch such project?

From now on, we consider one further complication, namely the hypothesis that the investment on the project has to be taken within a *finite* time T. We also assume that the value V_t is perfectly spanned by a liquid security X_t that is perfectly correlated to V_t. In a no arbitrage context the value of the project then takes the form

$$P(t, V_t; T) = \sup_{t \le \tau \le T} \mathbb{E}_t^Q \left[\left(e^{r(t-\tau)} V_\tau - I \right)^+ \right] \ , \tag{1}$$

where τ is a stopping time adapted to the Brownian's filtration and Q is an equivalent martingale measure chosen by the market and associated to the fact that we have a second source of uncertainty in the stochastic volatility.

A minute's thought reveals that the price $P(t, V_t; T)$ can be cast in terms of an *American* option with maturity T and a payoff $(X_\tau - I)^+$. In other words, an American call option. Due to the fact that the process X_t may have a drift under the measure Q distinct from the riskless interest rate r prevalent in the market the corresponding American option's optimal exercise time τ is not necessarily T. We are thus led to analyzing the problem of evaluating American call options on a dividend paying security under stochastic volatility. This problem, is known to have no explicit solutions in general. In order to analyze such problem we introduce the amply justified practical assumption of fast mean reversion [FPS00]. We remark in passing, that without such assumption one would have to resort to numerical techniques and handle the problem of determing the market price of volatility risk.

3. Stochastic Volatility Models under Fast Mean Reversion

In this section we focus on the case of *European* options and postpone the discussion of *American* options to Section 4. We recall the classical Black-Scholes (B-S) market model so as to fix the notation. We denote by ζ a riskless asset (bond or insured bank deposit) and by X a risky asset. In the classical B-S model the assets undergo the following dynamics

$$\mathrm{d}\zeta_t = r\zeta_t \mathrm{d}t \qquad \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \tag{2}$$

where W_t is the standard Brownian Motion. Let $P(t,x)$ denote the price of an option at time t and spot value x. Standard replication and non-arbitrage arguments lead to the classical Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + (r-\delta)x\frac{\partial P}{\partial x} - rP = 0 \qquad\qquad P(T_E,\cdot) = h \tag{3}$$

where h is the payoff at time T_E and δ is the continuous dividend rate.

As mentioned in Section 1, motivated by the need of explaining a number of empirical observations many authors consider stochastic volatility models. More precisely, following with [FPS00] and references therein, we consider the dynamics

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \qquad \sigma_t = f(Y_t) \qquad dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$$

where $\widehat{Z}_t$ is a linear combination of two independent Brownian motions (W_t) and (Z_t). As in [FPSS03b], we assume that f is bounded from above and away from zero. In addition to that we also assume f to be at least twice differentiable. In this model, the risky asset's volatility is controlled by a stochastic process $y = Y_t$, which could be thought of as a hidden process. Such process Y_t, in turn, undergoes an Ornstein-Uhlenbeck dynamics. This choice is motivated by the empirical remark that the volatility tends to return to a historical level after some time. The return rate to such mean is denoted by α.

Let $P = P(t,x,y)$ be the price of an European option at time t given that the current stock price is x and its driving state is y. Once again, using a non-arbitrage argument it is well-known [Hes93] that $P(t,x,y)$ satisfies

$$\begin{aligned} \frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} + (r-\delta)x\frac{\partial P}{\partial x} - rP + \\ \left(\alpha(m-y) - \beta\Lambda(t,x,y)\right)\frac{\partial P}{\partial y} = 0 \end{aligned} \tag{4}$$

where

$$\Lambda(t,x,y) = \rho\frac{\mu - r}{f(y)} + \gamma(t,x,y)\sqrt{1-\rho^2}$$

with final condition $P(T_E, \cdot, \cdot\cdot) = h(\cdot)$. Here, the function γ can be interpreted as the market value of risk associated to the second source of randomness that drives the volatility (Z_t). To avoid technical difficulties, we assume γ to be bounded and continuous. Furthermore, as in [FPS00], we shall assume that γ depends only on y.

Equation (4) can be interpreted, as done in [FPS00], considering the operator

$$\begin{aligned}\mathcal{L} =& \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2}{\partial x^2} + (r-\delta)x\frac{\partial}{\partial x} - r\cdot + \\ &+ \rho\beta x f(y)\frac{\partial^2}{\partial x\partial y} \\ &+ \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y} - \beta\Lambda\frac{\partial}{\partial y}\end{aligned}$$

The first line of the RHS for $\mathcal{L}$ consists of the standard Black-Scholes operator with (stochastic) volatility $f(y)$. The second one consists of a correlation term. The third one is the generator for the O-U process added to a premium term associated to the market price of volatility risk.

4. Main Results

We introduce the parameter $\epsilon = 1/\alpha$ where and consider the asymptotic behavior of the model when $\epsilon \to 0$. For the case of an American call option on a dividend paying asset, the Black-Scholes equation becomes the free-boundary value problem: given by

$$\begin{aligned}&\mathcal{L}^\epsilon P^\epsilon = 0, \\ &P^\epsilon(t,x,y) = (x-I)_+ \\ &P^\epsilon(t,x(t,y),y) = (x(t,y)-I)_+ \\ &\partial_x P^\epsilon(t,x^\epsilon_{\text{ex}}(t,y),y) = 1 \\ &\partial_y P^\epsilon(t,x^\epsilon_{\text{ex}}(t,y),y) = 0 \\ &x^\epsilon_{\text{ex}}(T,y) = I \\ &P^\epsilon(T,x,y) = (x-I)_+\end{aligned}$$

where $\mathcal{L}^\epsilon$ is the operator on the RHS of Equation 4 with $\alpha = 1/\epsilon$. We write

$$P^\epsilon = p_0 + \epsilon^{1/2}p_1 + \epsilon p_2 \quad x^\epsilon_{\text{ex}} = x_0 + \epsilon^{1/2}x_1 + \epsilon x_2 \ ,$$

and break the operator $\mathcal{L}^\epsilon$ into

$$\begin{aligned}
\mathcal{L}_0 =& \tilde{\nu}^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \ , \\
\mathcal{L}_1 =& \tilde{\nu}\rho\sqrt{2} x f(y) \frac{\partial^2}{\partial x \partial y} - \tilde{\nu} s(t,x,y)\frac{\partial}{\partial y} \ , \\
\mathcal{L}_2 =& \frac{\partial}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) - \delta x \frac{\partial}{\partial x} \ ,
\end{aligned}$$

$\tilde{\nu}^2 := \beta^2/(2\alpha)$, and $s(t,x,y) := (\beta/\alpha)\Lambda(t,x,y)$. After grouping the terms of equal order, proceeding with the straightforward analysis carried out in [FPS00] of the terms in ϵ^{-1} and $\epsilon^{-1/2}$, the relevant problems turn out to be

$$\begin{aligned}
&\mathcal{L}_2 P_0 = 0, x < x_0(t) \\
&P_0(t,x) = (x-I)_+, x > x_0(t) \\
&P_0(t, x_0(t)) = (x_0(t) - I)_+ \\
&\partial_x P_0(t, x_0(t)) = 1 \\
&x_0(T) = I.
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_2 P_1 = -\mathcal{L}_1 P_2, x < x_0(t) \\
&P_1(t,x) = 0, x > x_0(t) \\
&P_1(t, x_0(t)) = 0 \\
&x_1(t)\partial_x^2 P_0(t, x_0(t)) + \partial_x P_1(t, x_0(t)) = 0 \\
&x_1(T) = 0.
\end{aligned}$$

We computed the numerical solutions for the problems above, with $I = 50$, $r = \delta = 0.05$, $\sigma = 0.2$, $T = 2$, $V_1 = 0.25$, $V_2 = 0$. The data was chosen only for illustrative purposes.

The price of the American Call with dividends is given Figure 1. The price of the correction to allow for the stochastic volatility effects is shown in Figure 2.

5. Discussion

We have performed the analysis of an option to defer investment under a finite horizon assuming the presence of a spanning asset that satisfies a stochastic volatility model. The results presented in Figures 1 and 2 are typical of the results we obtained. They indicate a the significance of perturbation of the price, which seems to be always positive within the accuracy of the numerical software. Thus, the addition of the correction to the unperturbed solution increases the corresponding price of the option to defer and seems to increase also the optimal investment time. The intuitive reason for the increase in the solution is the fact that the extra source of uncertainty connected to the volatility seems to aggregate value to the firm's option to defer investment.

We have not attacked in this note the description of the free boundary that determines the exercise frontier. However, we expect that in this case due to the need of a multiscale analysis we would have to use the techniques developed in [SZ07b].

Acknowledgements

JPZ was supported by CNPq under grants 302161/2003-1 and 474085/2003-1.

References

[AM94] K Amin and A Morton. Implied volatility functions in arbitrage-free term structure models. *Journal of Financial Economics*, 35:141–180, 1994.

[AN93] K Amin and V Ng. Options valuation with systematic stochastic volatility. *Journal of Finance*, 48:881–909, 1993.

[CFPS04] Peter Cotton, Jean-Pierre Fouque, George Papanicolaou, and Ronnie Sircar. Stochastic volatility corrections for interest rate derivatives. *Math. Finance*, 14(2):173–200, 2004.

[Dix89] A. Dixit. Entry and exit decisions under uncertainty. *Journal Of Political Economy*, 97(3):620–638, June 1989.

[Dua95] J-C Duan. The garch option pricing model. *Mathematical Finance*, 5:13–32, 1995.

[Dup94] Bruno Dupire. Pricing with a smile. *Risk Magazine*, 7:18–20, 1994.

[FPS00] Jean-Pierre Fouque, George Papanicolaou, and K. Ronnie Sircar. *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press, 2000.

[FPS01a] Jean-Pierre Fouque, George Papanicolaou, and K. Ronnie Sircar. From the implied volatility skew to a robust correction to Black-Scholes American option prices. *Int. J. Theor. Appl. Finance*, 4(4):651–675, 2001.

[FPS01b] Jean-Pierre Fouque, George Papanicolaou, and Ronnie Sircar. Stochastic volatility and epsilon-martingale decomposition. In *Mathematical finance (Konstanz, 2000)*, Trends Math., pages 152–161. Birkhäuser, Basel, 2001.

[FPSS03a] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. Singular perturbations in option pricing. *SIAM J. Appl. Math.*, 63(5):1648–1665, 2003.

[FPSS03b] Jean-Pierre Fouque, George Papanicolaou, Ronnie Sircar, and Knut Solna. Multiscale stochastic volatility asymptotics. *Multiscale Model. Simul.*, 2(1):22–42 (electronic), 2003.

[Hes93] Steve Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[HPS92] N Hoffman, E Platen, and M Schweizer. Option pricing under incompleteness and stochastic volatility. *Mathematical Finance*, 2:153–187, 1992.

[HW87] J. Hull and A. White. The price of options on assets with stochastic volatilities. *J. Finances*, 42(2):281–300, 1987.

[IR92] J. E. Ingersoll and S. A. Ross. Waiting to invest - investment and uncertainty. *Journal Of Business*, 65(1):1–29, January 1992.

[MCC98] Dilip B. Madan, Peter P. Carr, and Eric C. Chang. The variance gamma process and option pricing. *Eur. Finance Rev.*, 2(1):79–105, 1998.

[MS86] R. MCdonald and D. Siegel. The value of waiting to invest. *Quarterly Journal Of Economics*, 101(4):707–727, November 1986.

[Mye77] S. C. Myers. Determinants of corporate borrowing. *Journal Of Financial Economics*, 5(2):147–175, 1977.

[Pin91] R. S. Pindyck. Irreversibility, uncertainty, and investment. *Journal Of Economic Literature*, 29(3):1110–1148, September 1991.

[PSS88] J. L. Paddock, D. R. Siegel, and J. L. Smith. Option valuation of claims on real assets - the case of offshore petroleum leases. *Quarterly Journal Of Economics*, 103(3):479–508, August 1988.

[RB94] A. Roma and C. Ball. Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis*, 29:589–607, 1994.

[RPT00] E. Renault, S. Pastorello, and N. Touzi. Statistical inference for random variance option pricing. *Journal of Business and Economic Statistics*, 18:358–367, 2000.

[RT96] Eric Renault and Nizar Touzi. Option hedging and implied volatilities in a stochastic volatility model. *Math. Finance*, 6(3):279–302, 1996.

[Sco87] L O Scott. Option pricing when the variance changes randomly: theory, estimation and an application. *Journal of Financial and Quantitative Analysis*, 22:419–439, 1987.

[SS91] E. Stein and J. Stein. Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies*, 4(4):727–752, 1991.

[SZ07a] Max O. Souza and Jorge P. Zubelli. On the asymptotics of fast mean-reversion stochastic volatility models. *Int. J. Theor. Appl. Finance*, 10(5):817–835, 2007.

[SZ07b] Max Oliveira Souza and J. P. Zubelli. Multiple scale asymptotics of fast mean reversion stochastic volatility models. In C. Fernandes, H. Schmidli, and N. Kolev, editors, *Third Brazilian Conference on Statistical Modelling in Insurance and Finance*, pages 248–253, São Paulo, 2007. IME-USP.

[Tit85] S. Titman. Urban land prices under uncertainty. *American Economic Review*, 75(3):505–514, 1985.

[TM87] L. Trigeorgis and S. P. Mason. Valuing managerial flexibility. *Midland Corporate Finance Journal*, 5(1):14–21, 1987.

[Tou79] O. Tourinho. The option value of reserves of natural resources. Working Paper 94, University of California, Berkeley, 1979.

[Tou99] N. Touzi. American options exercise boundary when the volatility changes randomly. *Applied Mathematics and Optimization*, 39:411–422, 1999.

[Wig87] J Wiggins. Options values under stochastic volatility. *J. Financial Economics*, 19(2):351–357, 1987.

[ZA98] Yingzi Zhu and Marco Avellaneda. A risk-neutral stochastic volatility model. *Int. J. Theor. Appl. Finance*, 1(2):289–310, 1998.

Departamento de Matemática Aplicada, Universidade Federal Fluminense, R. Mário Santos Braga, s/n, 22240-920, Niterói, RJ, Brasil.

IMPA, Est. D. Castorina 110, 22460-320, Rio de Janeiro, RJ, Brasil.

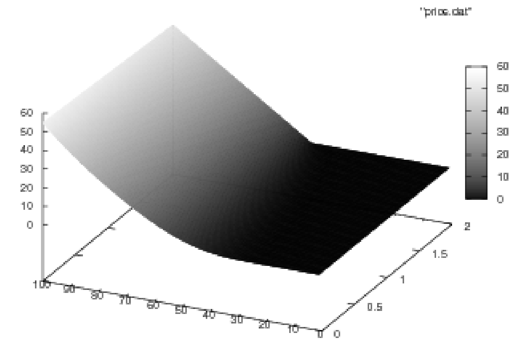

FIGURE 1. Black-Scholes price for the American Call with dividends

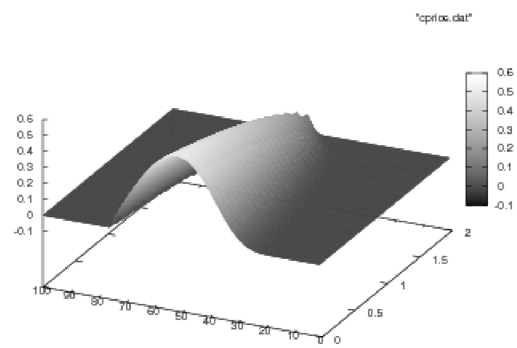

FIGURE 2. Stochastic volatility correction in the fast-mean reversion regime.