Strategic Investment Decisions under Fast Mean-Reversion Stochastic Volatility

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Abstract

We are concerned with investment decisions when the spanning asset that correlates with the investment value undergoes a stochastic volatility dynamics. The project value in this case corresponds to the value of an American call with dividends, which can be priced by solving a generalized Black-Scholes free boundary value problem. Following ideas of Fouque *et al.*, under the hypothesis of fast mean reversion, we obtain the formal asymptotic expansion of the project value and compute the adjustment of the price due to the stochastic volatility. We show that the presence of the stochastic volatility can alter the optimal time investment curve in a significative way, which in turn implies that caution should be taken with the assumption of constant volatility prevalent in many real option models. We also indicate how to calibrate to market data the model in the asymptotic regime.

1 Introduction

In uncertain times and under highly volatile market conditions the question of whether a corporate project should be delayed or started right away is a crucial one. Such uncertain times are usually associated to substantial changes in the market volatility levels. To handle strategic decisions, the use of *real option* techniques is by now well established [DP94, Tri96]. However, traditional real option analysis relies heavily on a constant volatility assumption. The latter is contradicted by any cursory look at market volatility data. Indeed, in many financial markets volatility tends to fluctuate at different levels and seems to mean-revert along a derivative contract life time. This led many authors to consider stochastic volatility market models [FPS00, Hes93, HW87, Wig87, SS91].

An additional feature of realized volatility is to mean revert, and in a quite fast rate. This is the so-called fast mean-reversion regime, and it has been observed in several markets with typical reversion lengths on the order of one to two days. See [FPS00] for data estimation in SP500 Market, and [SZ07b] for some data estimation on IBOVESPA. A detailed analysis of the IBOVESPA mean-reversion features, with pre-crisis data can be found in [Alv08]. The study of volatility of stochastic models in the fast mean-reversion regimes has received a great

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deal of attention in the recent financial literature after [FPS00]. See [CFPS04] for applications to interest rates, [FSS06] for applications to Defaultable Bonds, and [HJ08] for an application in commodity pricing.

In this article we pose and analyze the problem of strategic decisions such as the option to defer investment under fast mean-reversion volatility conditions. From the mathematical point of view it corresponds to a free-boundary partial differential equation associated to the option to delay an investment decision and to undertake it in the future [Mye77]. Such *real option* to defer investment has a value that can be modeled and quantified by option theory in the context of derivatives of *American type*. Such *option to defer* is associated, for example, to problems where management holds a lease on valuable land or resources and it can wait until output prices justify investment. See [MS86, PSS88, Tou79, Tit85, IR92]. For the real option approach we refer to [MS86, Dix89, TM87, Pin91]. We focus on the classical McDonald-Siegel [MS86] approach under the condition of stochastic volatility and fast mean reversion of the spanning asset.

We present the asymptotic analysis of the real option to defer investment for the McDonald-Siegel model [MS86] with the asset ongoing a stochastic volatility dynamics, under a fast mean reverting regime of stochastic volatility. We obtain, following [FPS00], the first order correction of the price and exemplify how this price could be used by means of numerical simulations. The latter indicate that the presence of stochastic volatility tends to add value to the option to defer and to increase the optimal time to start investment. This is particularly true when the option is near at the money. From a real option standpoint, this means that the analysis of borderline projects is the mostly likely to be affected by such a correction.

Both for didactic reasons and widespread use, we shall concentrate on the McDonald-Siegel model. Nevertheless, we remark that the analysis developed here can be extended to many other models, as for instance when the project value also mean-reverts. A large number of such models can be found, for example, in [DP94].

The plan for this article is the following: In Section 2 we briefly review the key aspects of the McDonald-Siegel model. In Section 3 we present the asymptotic expansion of the stochastic volatility model under fast mean reversion. Section 4 presents our main results concerning the asymptotic expansion under fast mean-reversion. We remark that such asymptotics is a formal one, and it opens the way for many important theoretical questions. Section 5 presents an example of calibration of the stochastic volatility parameters under fast mean reversion assumptions in the case of the IBOVESPA market index. In Section 6 we present our numerical results, draw some conclusions and describe some directions for further research.

2 The McDonald-Siegel Model

Suppose that a corporation is considering whether to launch a new project and let us assume that the estimated value of such project at a given time t is V_t. Suppose, furthermore that V_t evolves according to the stochastic differential equation $dV_t = \mu_\alpha V_t dt + \sigma_t V_t dW_t$, where here, differently from the traditional MS model, we take σ_t to be a stochastic process driven by a (hidden) stochastic process Y_t. The process Y_t, on the other hand, evolves according to a dynamics of the form $dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$, where Z_t is a Brownian possibly correlated to W_t.

Assume that the fixed cost of launching such project are known and given by I. Two fundamental questions appear:

- How much is such opportunity worth?
- What is the optimal time to launch such project?

From now on, we consider one further complication, namely the hypothesis that the investment on the project has to be taken within a *finite* time T. We also assume that the value V_t is perfectly spanned by a liquid security X_t that is perfectly correlated to V_t. In a no arbitrage context the value of the project then takes the form

$$P(t, V_t; T) = \sup_{t \le \tau \le T} \mathbb{E}_t^Q \left[\left(e^{r(t-\tau)} V_\tau - I \right)^+ \right] \,, \tag{1}$$

where τ is a stopping time adapted to the Brownian's filtration and Q is an equivalent martingale measure chosen by the market and associated to the fact that we have a second source of uncertainty in the stochastic volatility.

A minute's thought reveals that the price $P(t, V_t; T)$ can be cast in terms of an *American* option with maturity T and a payoff $(X_\tau - I)^+$ in the presence of dividends. Due to the presence of dividends and the fact that the process X_t may have a drift under the measure Q distinct from the riskless interest rate r prevalent in the market the corresponding American option's optimal exercise time τ is not necessarily T. We are thus led to analyzing the problem of evaluating American call options on a dividend paying security under stochastic volatility. This problem, is known to have no explicit solutions in general. In order to analyze such problem we introduce the amply justified practical assumption of fast mean reversion [FPS00]. We remark in passing, that without such assumption one would have to resort to numerical techniques and handle the problem of determining the market price of volatility risk.

3 Stochastic Volatility Models under Fast Mean Reversion

In this section we focus on the case of *European* options and postpone the discussion of *American* options to Section 4. We recall the classical Black-Scholes (B-S) market model so as to fix the notation. We denote by ζ a riskless asset (bond or insured bank deposit) and by X a risky asset. In the classical B-S model the assets undergo the following dynamics

$$\mathrm{d}\zeta_t = r\zeta_t \mathrm{d}t \qquad \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t \tag{2}$$

where W_t is the standard Brownian Motion. Let us assume that the asset X_t pays a continuous dividend at a rate δ and let $P(t, x)$ denote the price of an option at time t and spot value $X_t = x$. Standard replication and non-arbitrage arguments lead to the classical Black-Scholes equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 P}{\partial x^2} + (r - \delta)x\frac{\partial P}{\partial x} - rP = 0 \qquad\qquad P(T_E, \cdot) = h \tag{3}$$

where h is the payoff at time T_E and δ is the continuous dividend rate.

As mentioned in Section 1, motivated by the need of explaining a number of empirical observations many authors considered stochastic volatility models. More precisely, following [FPS00] and references therein, we consider the dynamics

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t \qquad \sigma_t = f(Y_t) \qquad dY_t = \alpha(m - Y_t)dt + \beta d\widehat{Z}_t$$

where $\widehat{Z}_t$ is a linear combination of two independent Brownian motions (W_t) and (Z_t). As in [FPSS03], we assume that f is bounded from above and away from zero. In addition to that we also assume f to be at least twice differentiable. In this model, the risky asset's volatility is controlled by a stochastic process $y = Y_t$, which could be thought of as a hidden process. Such process Y_t, in turn, undergoes an Ornstein-Uhlenbeck dynamics. This choice is motivated by the empirical remark that the volatility tends to return to a historical level after some time. The return rate to such mean is denoted by α.

Let $P = P(t, x, y)$ be the price of an European option at time t given that the current stock price is x and its driving state is y. Once again, using a non-arbitrage argument it is well-known [Hes93] that $P(t, x, y)$ satisfies

$$\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y) \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} + (r-\delta)x\frac{\partial P}{\partial x} - rP + \left(\alpha(m-y) - \beta\Lambda(t,x,y)\right)\frac{\partial P}{\partial y} = 0 \tag{4}$$

where

$$\Lambda(t,x,y) = \rho\frac{\mu - r}{f(y)} + \gamma(t,x,y)\sqrt{1-\rho^2}$$

with final condition $P(T_E, \cdot, \cdot\cdot) = h(\cdot)$. Here, the function γ can be interpreted as the market value of risk associated to the second source of randomness that drives the volatility (Z_t). To avoid technical difficulties, we assume γ to be bounded and continuous. Furthermore, as in [FPS00], we shall assume that γ depends only on y. Notice that [SZ07a] shows that γ cannot depend on x.

Equation (4) can be interpreted, as done in [FPS00], considering the operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + (r-\delta)x\frac{\partial}{\partial x} - r\cdot + \rho\beta x f(y)\frac{\partial^2}{\partial x \partial y} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y} - \beta\Lambda\frac{\partial}{\partial y}$$

The first line of the RHS for $\mathcal{L}$ consists of the standard Black-Scholes operator with (stochastic) volatility $f(y)$. The second one consists of a correlation term. The third one is the generator for the O-U process added to a premium term associated to the market price of volatility risk.

4 The Asymptotic Expansion

As mentioned above and substantiated by extensive empirical studies, the mean revertion rate α is large as compared to the characteristic time span under consideration. This leads naturally to the introduction of a small parameter $\epsilon = 1/\alpha$ and to consider the asymptotic behavior of the model when $\epsilon \to 0$. For the case of an American call option on a dividend paying asset, the Black-Scholes equation becomes the free-boundary value problem: given by

$$\begin{aligned}
&\mathcal{L}^\epsilon P^\epsilon = 0,\\
&P^\epsilon(t,x,y) = (x-I)_+\\
&P^\epsilon(t,x(t,y),y) = (x(t,y)-I)_+\\
&\partial_x P^\epsilon(t,x^\epsilon_{\text{ex}}(t,y),y) = 1\\
&\partial_y P^\epsilon(t,x^\epsilon_{\text{ex}}(t,y),y) = 0\\
&x^\epsilon_{\text{ex}}(T,y) = I\\
&P^\epsilon(T,x,y) = (x-I)_+
\end{aligned}$$

where $\mathcal{L}^\epsilon$ is the operator on the RHS of Equation 4 with $\alpha = 1/\epsilon$. We write

$$P^\epsilon = p_0 + \epsilon^{1/2} p_1 + \epsilon p_2 \quad x^\epsilon_{\text{ex}} = x_0 + \epsilon^{1/2} x_1 + \epsilon x_2 \ ,$$

and break the operator $\mathcal{L}^\epsilon$ into

$$\begin{aligned}
\mathcal{L}_0 =& \tilde{\nu}^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \ ,\\
\mathcal{L}_1 =& \tilde{\nu}\rho\sqrt{2} x f(y) \frac{\partial^2}{\partial x \partial y} - \tilde{\nu} s(t,x,y)\frac{\partial}{\partial y} \ ,\\
\mathcal{L}_2 =& \frac{\partial}{\partial t} + \frac{1}{2}\left(f(y)\right)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) - \delta x \frac{\partial}{\partial x} \ ,
\end{aligned}$$

$\tilde{\nu}^2 := \beta^2/(2\alpha)$, and $s(t,x,y) := (\beta/\alpha)\Lambda(t,x,y)$. After grouping the terms of equal order, proceeding with the analysis carried out in [FPS00] of the terms in ϵ^{-1} through $\epsilon^{1/2}$. After a long calculation and on invoking the appropriate solvability condition—as discussed in [FPS00], we find that the relevant problems turn out to be:

$$\begin{aligned}
&\mathcal{L}_2 P_0 = 0, x < x_0(t)\\
&P_0(t,x) = (x-I)_+, x > x_0(t)\\
&P_0(t,x_0(t)) = (x_0(t)-I)_+\\
&\partial_x P_0(t,x_0(t)) = 1\\
&x_0(T) = I.
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_2 P_1 = -\mathcal{L}_1 P_2, x < x_0(t)\\
&P_1(t,x) = 0, x > x_0(t)\\
&P_1(t,x_0(t)) = 0\\
&x_1(t)\partial_x^2 P_0(t,x_0(t)) + \partial_x P_1(t,x_0(t)) = 0\\
&x_1(T) = 0.
\end{aligned}$$

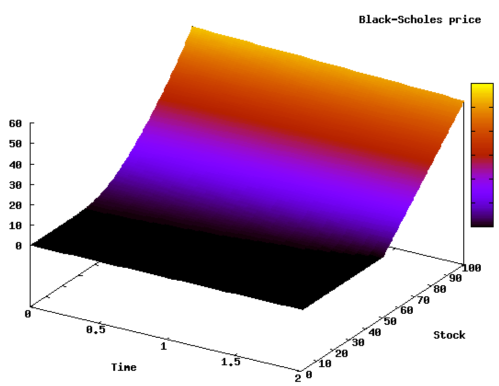
The expansion can then be written as

$$P(t,x,y) = P_{\text{BS}}(t,x;\bar{\sigma}) + \epsilon^{1/2}\left(T-t\right)\left[V_1 x^2 \frac{\partial^2 P_{\text{BS}}}{\partial x^2} + V_2 x^3 \frac{\partial^3 P_{\text{BS}}}{\partial x^3}\right],$$

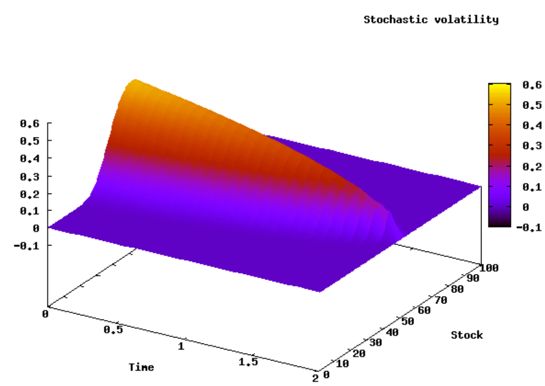
where P_{BS} is the Black & Scholes price for the American option, with effective volatility $\bar{\sigma}$. The constants V_1 and V_2 come from the calibration of the model to market data, and this is further discussed in the Section 5.

We computed the numerical solutions for the problems above, with $I = 50$, $r = \delta = 0.05$, $\sigma = 0.2$, $T = 2$, $V_1 = 0.25$, $V_2 = 0$. The data was chosen only for illustrative purposes.

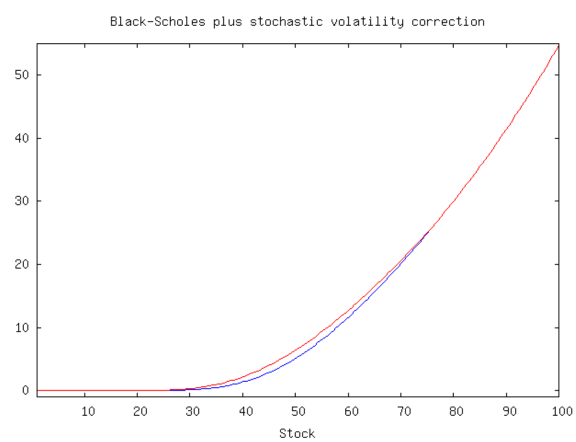
The price of the American Call with dividends is given Figure 1(a). The price of the correction to allow for the stochastic volatility effects is shown in Figure 1(b). Also, in Figure 1(c), there is a comparison between the Black-Scholes price and the Stochastic Volatility model.



(a) Black-Scholes price



(b) Correction



(c) B-S price versus B-S price plus stochastic correction

Figure 1: Comparison between prices in the Black-Scholes model and the stochastic volatility model.

5 An Example of Calibration from Real Data

Here we present some calibration details for the IBOVESPA index. We shall use historic data from the year 2006, but any other pre-crisis year with intra-day data available could be used for this purpose.

5.1 Fast Mean-Reversion

We begin by addressing one of the main hypothesis of the model, which is the fast mean-reversion of volatility.

Using historic data from 2006 and following the statistical procedure set by [FPS00], [Alv08] found a mean-reverting behavior, with

$$\alpha = 723 \quad \text{and} \quad \beta = 0.25.$$

This value of α implies a mean reversion time of approximately 1/3 of a day, which is indeed fast.

We also provide some additional information of the estimation procedure. The qualitative behavior of the IBOVESPA index and its returns can be seen in Figure 2

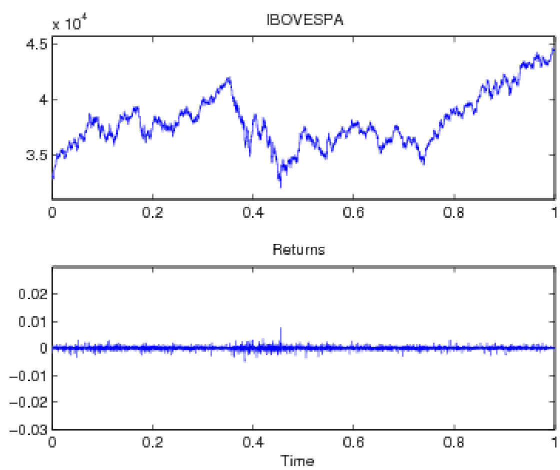


Figure 2: IBOVESPA index and returns in 2006.

The calibration procedure, as initially set by [FPS00] estimates the parameters from the decorrelation speed of an autocorrelogram or variogram. The variogram for 2006 can be seen in Figure 3. The oscillatory nature of the variogram is related to intra-day variation of volatility, and this was already noticed in [FPS00]. See also [FPSS04] for a discussion of this cycles from an implicit-volatility viewpoint. Finally, we can see the fitting of the model to the variogram in Figure 4.

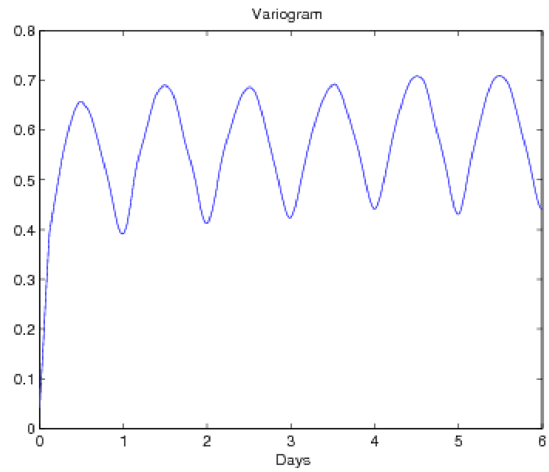


Figure 3: Variogram of IBOVESPA in 2006.

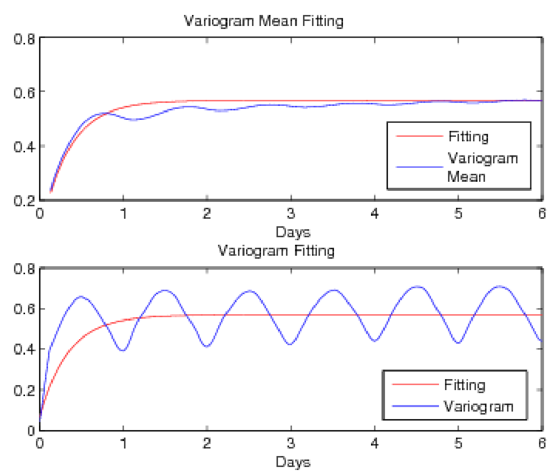


Figure 4: Variogram of IBOVESPA in 2006.

5.2 Calibration to Market Data

Following [FPS00], the asymptotic model is calibrated through the implied volatility surface. Under a fast mean-reversion assumption, the implied volatility for European options has the following asymptotic form

$$I = A\left[\frac{\log K/x}{T-t}\right] + B + \mathcal{O}(\epsilon), \tag{5}$$

where

$$A = -\frac{V_2}{\bar{\sigma}^3} \quad \text{and} \quad B = \bar{\sigma} + \frac{V_2}{\bar{\sigma}^3}\left(r + \frac{3}{2}\bar{\sigma}^2\right) - \frac{V_1}{\bar{\sigma}}. \tag{6}$$

Using such option price data, one can obtain a market price surface, which can be use to calibrate the model through the asymptotic representation of implied volatility (5); cf. [FPS00] and [Alv08]. This procedure has the advantage of implicitly calibrating the model to the market price of risk chosen by the market. This fitting can be seen in Figure 5. Once the

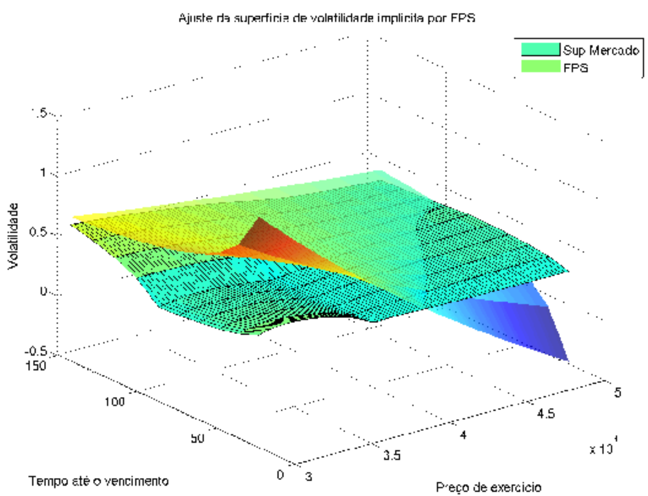


Figure 5: Implied vol fitting of the model at 11/04/2006.

coefficients A and B are obtained by the fitting procedure, once can obtain V_1 and V_2, and hence, we can fully complete the asymptotic stochastic volatility correction to the Black & Scholes price.

6 Discussion

We have performed the analysis of an option to defer investment under a finite horizon assuming the presence of a spanning asset that satisfies a stochastic volatility model. The results presented in Figures 1 are typical of the results we obtained. They indicate a significant perturbation of the price, which seems to be always positive within the accuracy of the numerical software. Thus, the addition of the correction to the unperturbed solution increases the corresponding price of the option to defer and seems to increase also the optimal investment time. The intuitive reason for the increase in the solution is the fact that the extra source of uncertainty connected to the volatility seems to aggregate value to the firm's option to defer investment. Additionally, we observe that the increase is larger for *at the money* prices. This also suggests that the value of optionality is enhanced when the project value is close to break-even, from the perspective of a Net Present Value analysis.

We have not attacked in this work the description of the free boundary that determines the exercise frontier. However, we expect that in this case due to the need of a multiscale analysis we would have to use the techniques developed in [SZ07a].

Acknowledgments

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