Towards a Theory of Chaotic Dynamics

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My aim today is to overview some of the effort that is currently being put into developing a theory of general chaotic dynamical systems. My lecture will have four main parts. Firstly, I shall formulate, in very general terms, the main problems in this field. Choosing an appropriate language to make this formulation more precise is a crucial step. This corresponds to the second part of the lecture, when I shall introduce a number of important notions. Next, I shall mention one class of systems, Hénon-like maps, for which the theory is already fairly complete, through results obtained in the last decade. The ultimate goal is even more ambitions: to build a global theory of systems with complex dynamical behaviour. In the last part of the lecture I mention some results, ideas and conjectures pointing in that direction.

1 General problems

The mathematical formulation of a dynamical system includes two ingredients: a set M, the *space of states* (usually a manifold with dimension $n \geq 1$), whose points represent the possible states of the system; and an *evolution law*, describing how the system evolves from one state to another. The latter may have *discrete-time*: a transformation $f\colon M \to M$ mapping each state $x_0 \in M$ to the one $x_1 = f(x_0)$ at which the system will be one unit of time later. The sequence $x_0, x_1, x_2, \dots$ defined by $x_{j+1} = f(x_j)$ is the *trajectory* of the *initial state* x_0. If the transformation f is invertible, then we also have a backward trajectory $\dots x_{-2}, x_{-1}, x_0$ defined by $x_{-j+1} = f(x_{-j})$. Another model is *continuous-time* evolution, expressed by a differential equation

$$\frac{dx}{dt} = F(x).$$

The trajectory of the initial state x_0 is the solution $x(t)$, $t \in \mathbb{R}$, of the differential equation with $x(0) = x_0$. It is assumed that solutions are defined for all times.

Problem 1. Describe the behaviour of most trajectories, for most dynamical systems, specially as time goes to infinity.

The mathematical formulation of the evolution law is always a simplification of the real process. So, it is very important that the conclusions drawn from it not be too specific: they should remain valid for nearby laws. That is what the next question is about:

Problem 2. Is the dynamical behaviour stable under small modifications of the evolution law (e.g. small variations of parameters involved in it) ?

2 Some fundamental notions

An *attractor* is a closed subset A of the state space M such that a large set of trajectories converge to A as time goes to infinity. More precisely this happens with *positive probability* when the initial state is picked at random. Mathematically speaking, what I mean is that the *basin of attraction* of A (the set of trajectories that converge to it) has positive volume inside the manifold M. As part of the definition of attractor one also requires *dynamical indecomposability*: there are trajectories dense in A, so that it can not be broken into smaller closed invariant pieces.

In simple terms, attractors tell us *where* typical trajectories go when time increases. One wants to know more: what is the spatial distribution of trajectories, that is, where do they spend more (or less) time?

Let me give a simple example. The state space is $\mathbb{R}$, and the evolution is described by $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2$. A simple computation shows that $f([-2,2]) = [-2,2]$: trajectories starting at some x_0 inside $[-2,2]$ remain in this interval for all future times. How do they behave? If one is interested in *all* the trajectories then there various types of behaviour. However, for typical (almost all) trajectories the answer is unique: they fill-in a dense subset of $[-2,2]$, i.e., each of them visits the neighborhood of any other state in this interval.

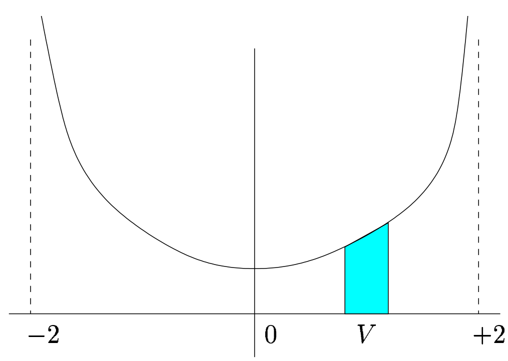

Figure 1: Distribution of a typical trajectory of $f(x) = x^2 - 2$

Are these typical trajectories evenly distributed in the whole $[-2,2]$? Figure 1 represents the graph of the distribution density: the integral over a subinterval V equals the fraction of time

$$\lim_{n\to\infty} \frac{1}{n} \#\{1 \le j \le n : x_j \in V\}$$

the trajectory spends in V. It shows that the distribuiton is not really uniform: comparatively, more time is spent near ± 2 than near zero. Actually, the distribution density is the same for all typical trajectories: if the initial state is chosen at random then, almost surely, it will generate the same graph as in the figure. This is rather remarkable indeed, because systems like this one are *sensitive with respect to the initial conditions* (or *chaotic*): trajectories of nearby initial states almost surely diverge from each other exponentially fast, becoming totally uncorrelated after a while.

Given any dynamical system, a *physical measure* is a probability $\mathcal{P}$ in the state space M such that

$$\mathcal{P}(V) = \text{fraction of time the trajectory of } x_0 \text{ spends in each domain } V \quad (1)$$

with positive probability (volume) when the initial state x_0 is picked at random. In the example I discussed before $\mathcal{P}$ is Lebesgue measure on $[-2, 2]$, multiplied by the density in Figure 1, and (1) holds with full probability for x_0 in $[-2, 2]$. The notion of physical measure goes back to Sinai, Ruelle, Bowen [Sin72], [Rue76], [Bow75], who proved that uniformly hyperbolic systems (Axiom A systems, as defined by Smale [Sma67]) have a finite number of physical measures (or SRB measures). The theory of uniformly hyperbolic systems is a good paradigm for what we would like to have in much more generality.

Before stating some rigorous results, let me briefly discuss Problem 2. I am specially interested in *stability under small random perturbations*. The following situation is often encountered in practical applications. The transformation $f\colon M \to M$ is a simplification of the actual dynamics, taking into account only the main aspects of the evolution: other aspects are too small and/or too complex to go into the mathematical formulation of the problem. They may, however, affect the dynamics to some extent: from x_0 the system moves on to a state $\tilde{x}_1$ which is close, but not quite equal to $x_1 = f(x_0)$. Then it moves to a state $\tilde{x}_2$ close to $f(\tilde{x}_1)$, and so on. In many cases, one may think of each $\tilde{x}_{j+1}$ as resulting from $f(\tilde{x}_j)$ by the addition of a small amount of random noise (independent at each iteration step). The question is: is the information one gets from the simplified model $f\colon M \to M$ compatible with the "true" behaviour, described by the sequence $(\tilde{x}_j)_j$?

Let me be more precise. Out of general results, there exists a probability $\mathcal{P}_\varepsilon$ (ε denotes the size of the random noise) describing the behaviour of typical pseudo-trajectories $(\tilde{x}_j)_j$:

$$\mathcal{P}_\varepsilon(V) = \text{fraction of time for which } \tilde{x}_j \text{ is in each domain } V \text{ in } M$$

with positive probability in the choice of $\tilde{x}_0 = x_0$, and full probability in the choice of the subsequent $\tilde{x}_j$. The system is *stochastically stable* if $\mathcal{P}_\varepsilon$ is close to $\mathcal{P}$ when the random noise is small. See e.g. [Via97] for more detailed information.

3 Hénon-like systems

The model $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (1 - ax^2 + y, bx)$, was proposed by Hénon [Hén76], as the simplest mathematical formulation for a system with complex dynamics. He observed, numerically, the presence of a chaotic (or "stange") attractor for parameters $a \approx 1.4$, $b \approx 0.3$,. Starting from the work of Benedicks, Carleson [BC91], after the pioneer results of Jakobson [Jak81], it has been possible to develop a rather complete theory for this class of maps (although we do not quite understand the particular parameter range studied by Hénon yet). This theory is summarized in the theorem below. We also know that it applies to very general situations in Dynamics [MV93], [DRV96].

Theorem 1. *With positive Lebesgue probability in the space of parameters a and b,*

(a) f has a chaotic attractor A (Benedicks, Carleson [BC91]);

(b) f has a physical measure $\mathcal{P}$ (Benedicks, Young [BY93]);

(c) there are no-holes in the basin of attraction of A (Benedicks, Viana [BVb]);

(d) the chaotic attractor is stochastically stable (Benedicks, Viana [BVa]).

Part (c) means that (1) holds not only with positive probability but, in fact, with full (area) probability in the basin of attraction.

4 Towards a global theory

The following conjectures are part of Palis' program, presented in [Pal99]:

Conjecture 1. Any system can be approximated (small modification of the evolution law) by another having only a finite number of attractors which, moreover, have nice properties: physical measures, the no-holes property, stochastic stability.

Conjecture 2. Most systems, in terms of probability in parameter space, have finitely many attractors.

In this direction we have

Theorem 2 (Alves, Bonatti, Viana [ABV]). *Every strongly chaotic transformation has a finite number of attractors and physical measures.*

The precise definition of *strongly chaotic* is somewhat technical. I just mention a consequence, and refer the reader to [ABV] for the complete formulation. A strongly chaotic map is sensitive in all directions: given $x_0 \in M$ and any y_0 close to it, almost surely, the corresponding trajectores diverge from each other in the future, exponentially fast. More precisely, all the Lyapunov exponents are positive, almost everywhere.

This result motivates the following conjecture that I formulated for the first time in [Via98].

Conjecture 3. Any smooth system whose Lyapunov exponents are non-zero almost everywhere (with respect to volume) has physical measures (a finite number of them if the Lyapunov exponents are bounded from zero).

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