A stochastic view of Dynamical Systems

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Dynamical systems

Transformations $f : M \to M$ (discrete time)

or flows $f^t : M \to M$ (continuous time)

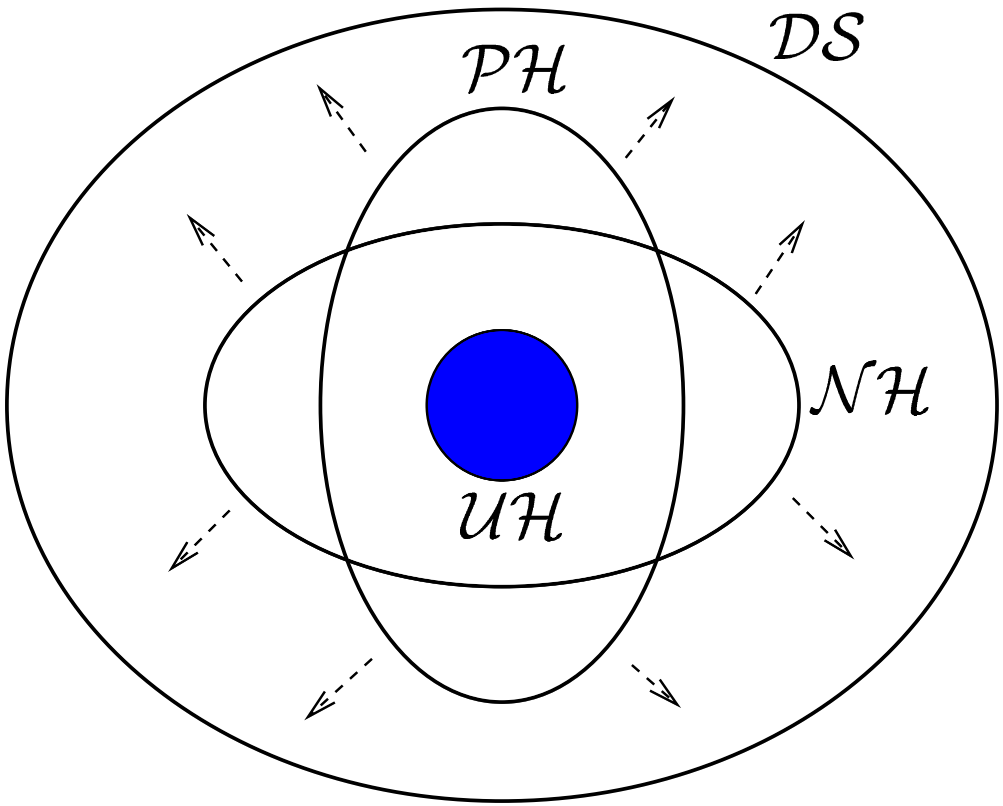
on some state space M.

General goals: for “most” systems and “most” initial states,

- describe the dynamical behavior, and
- analyze its stability under perturbations

There has been considerable progress towards a global theory, with a strong stochastic flavor.

Towards a global theory



$\mathcal{UH}$= uniformly hyperbolic systems

$\mathcal{PH}$= partially hyperbolic $\mathcal{NH}$= (non-uniformly) hyperbolic

Uniformly hyperbolic systems

Consider the *cat map*

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2.$$

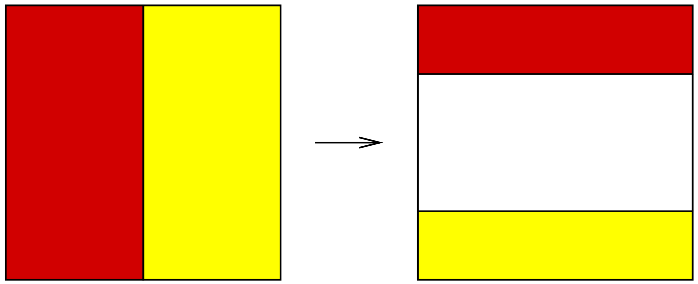
For almost every initial state $z \in \mathbb{T}^2$, and any observable (continuous function) $\varphi : M \to \mathbb{R}$,

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(z)) \to \int \varphi \, dm$$

where $m =$ volume (Lebesgue) measure on $\mathbb{T}^2$.

Uniformly hyperbolic systems

A similar fact is true, but much more subtle, for the *slim baker map* (or any perturbation of it):



There exists an invariant probability μ such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) \to \int \varphi\, d\mu$$

for almost every initial state $z \in \mathbb{T}^2$, and any observable $\varphi : M \to \mathbb{R}$.

Physical measures

A *physical measure* is an invariant probability μ in the state space such that the set $B(\mu)$ of initial states z such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(z)) \to \int \varphi\, d\mu$$

for any observable $\varphi : M \to \mathbb{R}$, has (at least) positive volume.

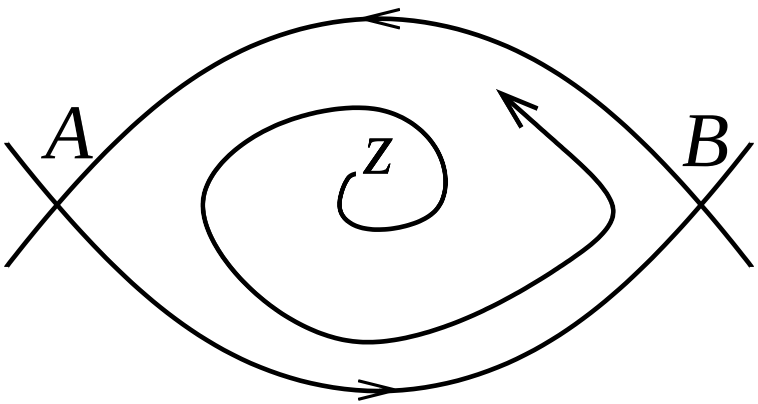
This means that, for any $z \in B(\mu)$ and typical sets $V \subset M$,

$\mu(V) =$ average time the orbit of $z \in B(\mu)$ spends in V.

Physical measures

- Existence of physical measures ?
- Uniqueness/Finiteness ?

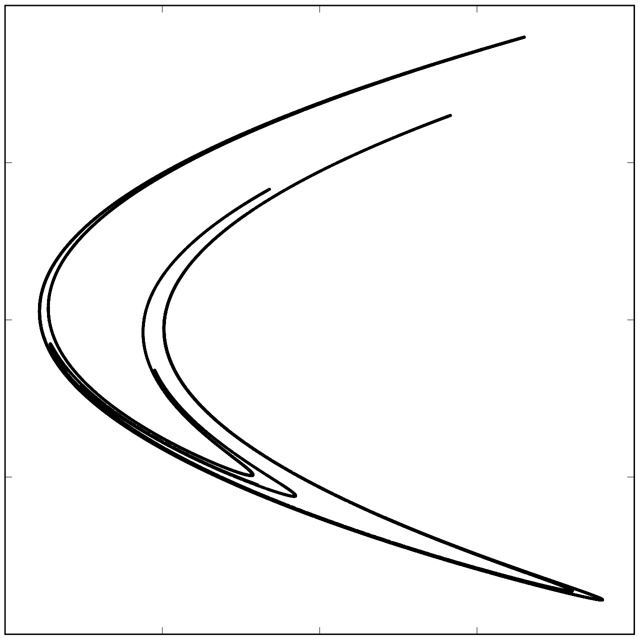
These problems are hard. Time averages need not converge. For example:



For uniformly hyperbolic systems there is a very complete theory, by Sinai, Ruelle, Bowen (1970-1976).

Hénon strange attractors

$f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (1 - ax^2 + y, bx)$



The picture is for paarmeters $a = 1.4$ and $b = 0.3$.

Hénon strange attractors

Benedicks-Carleson (1991): for a positive probability set of parameters, there exists a *strange attractor*.

Benedicks-Young (1993,96): the attractor supports a physical measure μ, for which the system has exponential loss of memory (decay of correlations).

Benedicks-Viana (2000): the physical measure μ is unique, and $B(\mu)$ contains almost all points in the basin of attraction.

strange $\sim$ *exponentially sensitive* $\sim$ *(non-unif) hyperbolic*

all Lyapunov exponents are non-zero at almost all points.

Physical measures

Conjecture: If all Lyapunov exponents are non-zero almost everywhere (full volume), then almost every point is contained in the basin of some physical measure.

There are partial results by Alves, Bonatti, Viana and by Pinheiro.

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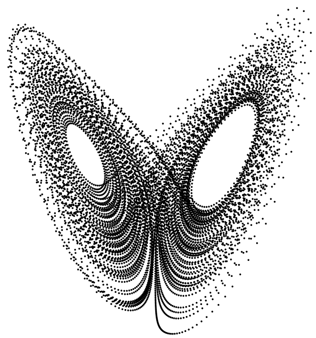
Conjecture (Palis): Any system may be approximated by another for which there are only finitely many physical measures, and the union of their basins has full volume.

It is known to be true for transformations in dimension 1, by Lyubich and Avila, Moreira.

Physical measures

There is a good understanding of other models, including

- Lorenz strange attractors, by Morales, Pacifico, Pujals, Tucker.



- partially hyperbolic maps, by Alves, Bonatti, Viana and by Tsujii.

Partial hyperbolicity

Partial hyperbolicity: in addition to exponentially expanding and exponentially contracting directions, one allows for *central* directions, where the behavior may be neutral, or else vary from one point to the other.

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There has been much progress in partially hyperbolic systems, both conservative and dissipative:

Pugh, Shub, Wilkinson, Burns, Nitika, Torok, Xia, Arbieto, Matheus, Tahzibi, Horita,

Diaz, Pujals, Ures, Bonatti, Viana, Arnaud, Wen, Crovisier, Abdenur, and others.

Stochastic stability

Consider the stochastic process obtained by adding small random noise to the system. For discrete time, there are two main models (most results hold for both):

- (Markov chain) $z_{n+1} = f(z_n) + \varepsilon_n$ with $\varepsilon_n \in B_\varepsilon(0)$
- (random maps) $z_{n+1} = f_n(z_n)$ with $f_n \in B_\varepsilon(f)$

Under general assumptions, there exists a stationary probability μ_ε such that, almost surely,

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(z_j) \to \int \varphi \, d\mu_\varepsilon.$$

Stochastic stability

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Conjecture: If all Lyapunov exponents are non-zero almost everywhere, and there is a unique physical measure, then the system is stochastically stable.