Abundance of stable ergodicity

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Abstract

We consider the set $\mathcal{PH}_\omega(M)$ of volume preserving partially hyperbolic diffeomorphisms on a compact manifold having 1-dimensional center bundle. We show that the volume measure is ergodic, and even Bernoulli, for any C^2 diffeomorphism in an open and dense subset of $\mathcal{PH}_\omega(M)$. This solves a conjecture of Pugh and Shub, in this setting.

To Charles and Mike: Happy 60th birthdays!

1 History

A fundamental problem, going back to Boltzmann and the foundation of the kinetic theory of gases, is to decide how frequently conservative dynamical systems are ergodic.

A first striking answer was provided by KAM (Kolmogorov, Arnold, Moser) theory: ergodicity is not a generic property, in fact there are open sets of conservative systems exhibiting positive volume sets consisting of invariant tori supporting minimal translations.

In sharp contrast with this elliptic type of behavior, ergodicity prevails at the other end of the spectrum, namely, among strongly hyperbolic systems. Indeed, after partial results of Hopf and Hedlund, Anosov proved that the geodesic flow of any compact manifold with negative curvature is ergodic.

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In fact, the same is true for any sufficiently smooth conservative uniformly hyperbolic flow or diffeomorphism.

By the mid-nineties, Pugh and Shub proposed to address the ergodicity problem in the context of partially hyperbolic systems, where the tangent space splits into uniformly contracting (stable), uniformly expanding (unstable), and "neutral" (central) directions. To summarize their main theme:

A little hyperbolicity goes a long way in guaranteeing ergodicity.

In more precise terms, in [11] they proposed the following

Conjecture. *Stable ergodicity is a dense property among C^2 volume preserving partially hyperbolic diffeomorphisms.*

At about the same time, there was a renewed interest in the geometric and ergodic properties of partially hyperbolic systems in the broader context of possibly non-conservative dynamical systems. A main goal here was to establish existence and finiteness of SRB (Sinai, Ruelle, Bowen) measures, and to characterize their basins of attraction.

Thus the general theme of partially hyperbolic dynamics evolved into a very active research field, with contributions from a large number of mathematicians. See, for instance, [2, 6] for detailed accounts of much progress attained in the last few years.

2 Result

The purpose of this note is to point out that, putting together recent results by Shub, Wilkinson [12] followed by Baraviera, Bonatti [1], by Bonatti, Viana [3] followed by Burns, Dolgopyat, Pesin [5], and by Dolgopyat, Wilkinson [7], one obtains a proof of the conjecture stated above, when the central direction is 1-dimensional.

Theorem. *Let M be a compact manifold endowed with a smooth volume form ω, and $\mathcal{PH}_\omega(M)$ be the set of all partially hyperbolic diffeomorphisms having 1-dimensional center bundle and preserving the volume form.*

Then the volume measure defined by ω is ergodic, and even Bernoulli, for any C^2 diffeomorphisms in a C^1 open and dense subset of $\mathcal{PH}_\omega(M)$.

The proof of the theorem follows. In fact, we prove a bit more: every C^2 diffeomorphism in $\mathcal{PH}_\omega(M)$ is C^1 approximated by another C^2 diffeomorphism in $\mathcal{PH}_\omega(M)$ which is stably Bernoulli. Note that it is not known whether C^2 maps are dense in $\mathcal{PH}_\omega(M)$.

Throughout, all maps are assumed to be volume preserving. First, [1] extends the technique of [12], to prove that every partially hyperbolic diffeomorphism may be C^1 approximated by another for which the integrated sum of all Lyapunov exponents along the central direction is non-zero. Under our dimension assumption, this just means that the integrated central Lyapunov exponent is non-zero, for a C^1 open and dense subset $\mathcal{O}_1$ of partially hyperbolic diffeomorphisms. Let us decompose $\mathcal{O}_1$ as $\mathcal{O}_- \cup \mathcal{O}_+$, according to whether the integrated central exponent is negative or positive.

Up to replacing f by its inverse, we may suppose that $f \in \mathcal{O}_-$. For such f, there is a positive volume set of points with negative central Lyapunov exponent. Assuming f is C^2, the arguments in [3] show that there exists an invariant ergodic Gibbs u-state μ with negative central Lyapunov exponent. See the Lemma below for a proof. Then μ is an SRB measure and, as observed in [5], its basin contains a full volume measure subset of some open set $O(\mu)$, which is saturated by both strong foliations.

Also for f in a C^1 open and dense subset $\mathcal{O}_2$, [7] proves that the diffeomorphism has the accessibility property: any two points may be joined by a path formed by finitely many segments contained in leaves of the strong-stable foliation or the strong-unstable foliation. Taking f of class C^2 in $\mathcal{O}_- \cap \mathcal{O}_2$ we obtain that $O(\mu)$ is the whole manifold so that the basin of μ has total volume in M. This implies ergodicity.

Finally, the same arguments extend directly to any iterate f^n, $n \geq 1$. Indeed, $f^n \in \mathcal{O}_\pm$ if and only if $f \in \mathcal{O}_\pm$, and μ is an SRB-measure also for f^n. Moreover, f^n is accessible if and only if f is, since the two maps have the same strong foliations. This shows that f^n is ergodic, for every $n \geq 1$, whenever $f \in \mathcal{O}_\pm \cap \mathcal{O}_2$. Using Theorem 8.1 of Pesin [10], we conclude that f is Bernoulli.

3 Conclusion

To conclude, we give the technical definitions of the notions involved, and we state and prove the Lemma.

Let M be a compact manifold endowed with a volume form ω. A volume preserving diffeomorphism $f : M \to M$ is *stably ergodic* if the volume measure defined by ω is ergodic for any C^2 diffeomorphism in a C^1-neighborhood of f.

A diffeomorphism $f : M \to M$ is *partially hyperbolic* if there is a splitting $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle into three invariant bundles (with positive dimension) and there exists $m \geq 1$ such that

$$\|Df^m \mid E^s\| \leq \frac{1}{2} \quad \text{and} \quad \|Df^{-m} \mid E^u\| \leq \frac{1}{2}$$

and

$$\|Df^m \mid E^s\| \, \|(Df^m \mid E^c)^{-1}\| \leq \frac{1}{2} \ \text{ and } \ \|(Df^m \mid E^u)^{-1}\| \, \|Df^m \mid E^c\| \leq \frac{1}{2}\,.$$

The first condition means that E^s is uniformly contracting and E^u is uniformly expanding. The last one means that the splitting is *dominated.*

We denote $\mathcal{PH}(M)$ the space of partially hyperbolic C^1 diffeomorphisms on M with $\dim E^c = 1$, and $\mathcal{PH}_\omega(M)$ the subset of volume preserving diffeomorphisms.

Let $f \in \mathcal{PH}(M)$. Then the stable bundle E^s and the unstable bundle E^u are uniquely integrable. The corresponding integral foliations, respectively strong-stable $\mathcal{F}^s$ and strong-unstable $\mathcal{F}^u$ are invariant, and their leaves are uniformly contracted by all forward and backward iterates of f, respectively.

We say that $f \in \mathcal{PH}(M)$ has the accessibility property if any two points of M may be joined by a path formed by finitely many segments contained in leaves of the strong-stable foliation or the strong-unstable foliation.

A *Gibbs u-state* is an invariant probability with absolutely continuous conditional measures along the leaves of the strong-unstable foliation. Gibbs s-states are defined in the same fashion. In the partially hyperbolic context, such measures were first constructed by Pesin, Sinai [9].

An invariant probability measure μ is an *SRB measure* if the set of points $x \in M$ whose time averages

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x))} \to \mu \qquad \text{(weakly)}$$

has positive volume. This set is called the *basin* of μ.

Recall that $\mathcal{O}_-$ denotes the subset of diffeomorphisms in $\mathcal{O}$ with negative integrated central Lyapunov exponent. The following lemma is essentially contained in [3]:

Lemma. *Any C^2 diffeomorphism $f \in \mathcal{O}_-$ has some Gibbs state with negative central Lyapunov exponent.*

Proof. Let $f \in \mathcal{O}_-$ be a C^2 diffeomorphism. Denote by $\mathcal{R}$ the set of points $x \in M$ for which the Lyapunov exponent

$$\lambda^c(x) = \lim_{n\to\pm\infty} \frac{1}{n} \log |Df^n \mid E^c_x|$$

is well defined; that is, both limits exist and they coincide. As f preserves volume, the ergodic theorem ensures that $\mathcal{R}$ has full volume in M. Since the strong-unstable foliation $\mathcal{F}^u$ is absolutely continuous [4], there is a full measure subset $\mathcal{R}_0$ of points $x \in \mathcal{R}$ for which the intersection $\mathcal{R} \cap \mathcal{F}^u(x)$ has full Lebesgue measure inside $\mathcal{F}^u(x)$, where $\mathcal{F}^u(x)$ denotes the strong-unstable leaf through x.

On the other hand, the hypothesis

$$\int_M \lambda^c(x)\, d\omega(x) = \int_M \log |Df \mid E^c_x|\, d\omega(x) < 0$$

implies that there exists a positive volume set $\mathcal{R}^-$ of points x such that $\lambda^c(x)$ is negative. Observe that $\lambda^c(x) = \lambda^c(y)$ if x, $y \in \mathcal{R}$ belong to the same strong-unstable leaf. Choose $x_0 \in \mathcal{R}^- \cap \mathcal{R}_0$ and let $D \subset \mathcal{F}^u(x)$ be a disk centered at x. Let m_0 be the normalized Lebesgue measure induced on D by some Riemannian metric of M. Then m_0 is a probability measure, and $m_0(D \cap \mathcal{R}^-) = 1$. Let

$$m_n = \frac{1}{n} \sum_0^{n-1} f^i_*(m_0).$$

By [9], every accumulation point μ of the sequence m_n is a Gibbs u-state for f. Moreover, since λ^c is well defined and equal to $\lambda^c(x_0)$ at m_0-almost every point,

$$\int_M \log |Df \mid E^c_x|\, d\mu(x) = \lim_{n \to +\infty} \int_M \log |Df \mid E^c_x|\, dm_n(x) = \lambda^c(x_0) < 0.$$

Then at least one ergodic component μ_0 of μ must have

$$\int_M \log |Df \mid E^c_x|\, d\mu_0(x) \leq \lambda^c(x_0) < 0.$$

Finally, [3] asserts that each ergodic component of a Gibbs u-state is again a Gibbs u-state. Hence, μ_0 is the announced ergodic Gibbs u-state with negative central Lyapunov exponent. □

4 Questions

One would like to remove the assumption on the central dimension.

Another important open problem is the C^r version of the conjecture, any $r > 1$. In this direction, Niţică, Török [8] prove C^r density of accessibility assuming a r-normally hyperbolic 1-dimensional, integrable central bundle with at least two compact leaves.

Here we prove ergodicity assuming C^2 regularity. While ergodic systems always form a G_δ, it is not known whether C^2 maps are dense in the space C^1 volume preserving diffeomorphisms; see Zehnder [13]. So it remains open whether ergodicity is generic (dense G_δ) among C^1 partially hyperbolic with 1-dimensional central bundle.

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